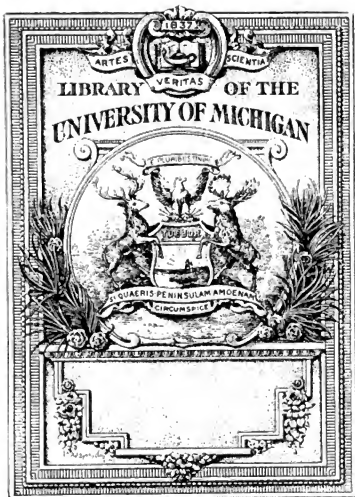




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THE  
ELEMENTS  
OF  
ALGEBRA.

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THE  
ELEMENTS  
OF  
ALGEBRA

IN A  
New and Easy Method;

WITH THEIR  
USE and APPLICATION,  
IN THE

Solution of a great Variety of Arithmetical  
and Geometrical Questions;

By general and universal RULES.

To which is prefixed an

INTRODUCTION,  
CONTAINING

A Succinct HISTORY of this SCIENCE.

---

By *NATHANIEL HAMMOND*,  
Of the BANK.

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The FOURTH EDITION, Corrected.

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L O N D O N,

Printed for the AUTHOR:

And Sold by E. and C. DILLY, in the *Poultry*; and  
W. DOMVILLE, under the *Royal Exchange*.

MDCCLXXII.

There are two Things absolutely necessary, to make the Acquisition of any Science as easy as its Nature will admit. First, the Disposition of the Work, so that the Rules be clear and distinct; and then the Illustration of these Rules by a sufficient Number of proper and pertinent Examples. And tho' the excellent *Elements* of the judicious EUCLID are of a different Nature, yet in this I have industriously studied to imitate him, that I propose no new Rule or Article in this *Science of Investigation*, till it become necessary to carry the Learner to a further Degree of Knowledge.

Science may be compared to a highly finished Pile of Building, all the Parts of which being disposed in the most exact Symmetry, they must affect our Perception, and gratify our internal Sensation with a more exquisite Pleasure, than if viewed in a separate State: For in such a State, to all but the Learned, they would appear broken and incon- nected Materials of a mighty Structure, which the Mind wanting Power to conceive, could enjoy no Satisfaction in the Contemplation of such a Train of imperfect and confused Ideas. But, when thus exhibited in their true Proportion, it will be easy, even for the youngest Scholar, to gain a perfect Notion of each, and, as he advances, a gradual Comprehension of the Beauty resulting from their Connection, and how they mutually assist and ornament each other.

In teaching the abstract Sciences, Examples have a strong as well as natural Tendency to illustrate the Precepts concerning our abstract Reasonings, especially in this Science of *Investigation*: In the *Synthetic* Method, or Method by Demonstration, one  
Example

Example or Proposition is sufficient, for a Number of the same Propositions is only a Repetition of the same successive Train of Ideas; whereas, in the *Analytic* Method, or Method by *Investigation*, the same Conclusion is gained, tho' there may be a great Variety in the Connection of the Reasoning, by the different Disposition of the Ideas, notwithstanding they are directed by the same general Rule.

And as the principal Difficulty in this Science, is acquiring the Knowledge of solving of Questions, I have given a great Variety of these in respect to Numbers and Geometry, and their Solutions I chose to give in the most particular, distinct, and plain Manner; and for which the Reader will find full and explicit Directions. As it is preposterous and absurd, to expect a Person to be a *Critic* in any Language as soon as he has passed thro' his Grammar; so, I cannot help thinking it wrong to expect a Learner should see the Reason of particular elegant Methods of Solution, before he has practised a general and universal Method; but after the general Rules are become easy and familiar, the Learner may then apply himself to the particular Methods. And I know of no Work that has illustrated and exemplified the general and universal Rules in so copious a Manner, as will be found in the following Sheets.

In the Arithmetical Operations, the Decimal Fractions are continued to two Places only, these being sufficient to shew the Reader, that if the Question admits not of an exact Answer, he is yet near the Truth, and may prosecute the Answer to any required Degree of Exactness. I have avoided all tedious Numerical Calculations, as they have no  
Tendency

Tendency to increase the Learner's Knowledge in *Algebra*.

The Rules of Vulgar Fractions in *Algebra* are omitted, being generally very perplexing to Learners; but as I have given sufficient Directions how they are managed whenever they occur in the Solution of any Question, the Reader will find no Difficulty in reducing an Equation with Fractional Quantities.

It is necessary my Reader should understand Vulgar and Decimal Fractions in common Arithmetick, and the Extraction of the Square Root, and then I know no Reason why a Person may not make himself a perfect Master of the following Work, excepting the Geometrical Questions, which he may omit, and proceed to those which require no Skill in Geometry; for thro' the whole, where it was necessary, I have given the same Directions, as if I was actually teaching a Scholar.

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T H E

# INTRODUCTION.

**A**S all Arts have their Beginning, rude and weak, and reach Perfection by Degrees, so that, which is the Subject of the following Sheets, has been cultivated by so many illustrious Men in our own, as well as in foreign Nations, that it cannot but appear a natural Introduction to this Treatise, if we digest the History of its Rise and Progress into a succinct Discourse; the rather, because Books of this Sort are now become very numerous in ours, as well as in other Languages, and therefore, it is the more necessary to record the Names of such as have eminently improved so useful a Branch of Knowledge.

The Word *Algebra* is certainly derived from the *Arabic*, but there have been some Mistakes as to its Meaning. When it was first introduced in *Europe*, it was understood to be the Invention of the famous Philosopher *Geber*; and therefore *Michael Stifelius* calls it sometimes *Regula Algebrae*, and sometimes *Regula Gebri*, whence it is plain he understood by it no more than the Rule of *Geber*, or, as we usually express it, *Geber's Rule*. But when we became better acquainted with *Arabic Learning*, this Derivation appeared ill founded: In that Language, this Art is called *Al-gjábr W'al-mokábala*, which is literally, the Art of Resolution and Equation. Hence it is plain, we had the Word *Algebra* from the *Arabic* Name of the Art, and not from the pretended Inventor. But it may not be amiss to observe, that the *Arabic* Name contains a Definition, or is rather an emphatic Declaration

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ration of the Nature and End of this Science; for the *Arabic* Verb *jábara* signifies to reset, and is properly used in respect to Dislocations, and the Verb *bábala*, implies opposing, or comparing; and how applicable this is to what we call Algebra, the Reader, when he is thoroughly acquainted with this Book, will easily understand. As it became better known to the *Europeans*, it received different Names; the *Italians* stiled it *Ars magna*, in their own Language *l'Arte Magjore*, opposing to it common Arithmetick, as the lesser or minor Art. It was also called *Regula Cosæ*, the Rule of *Cosæ*, for an odd Reason: The *Italians* make use of the Word *Cosæ*, to signify what we call the Root, and from thence, this Kind of Learning being derived to us from them, the Root, the Square, and the Cube, were called *Cosfick* Numbers, and this Science the Rule of *Cosfæ*. I should not have dwelt so long on so dry a Subject, but that it is absolutely necessary for the understanding what follows.

It is a Point still disputed, whether the Invention of Algebra ought to be ascribed to the *Oriental* Philosophers, or to the *Greeks*; but it is a Thing certain, that we received it from the *Moors*, who had it from the *Arabians*, who own themselves indebted for it to the *Persians* and *Indians*; and yet, which is strange enough, the *Persians* refer the Invention to the *Greeks*, and particularly to *Aristotle*. Yet, notwithstanding this, it must be allowed that the Algebra taught us by the *Arabians* differs very much from that contained in the Works of *Diophantus*, the eldest *Greek* Author on this Art, which is now extant, and which was discovered and published long after the Algebra taught by the *Arabians* had been studied and improved in the West. But all these Difficulties, which have given some great Men so much Trouble, may be easily surmounted, if we suppose that the Invention was originally taken from the *Greeks*, and new modelled by the *Arabians*, in the same Manner as we know that common Arithmetick was; for this, which is at least extremely probable, makes the whole plain and  
clear,

clear, and leaves us at liberty to pursue the Progress of this Art from the first printed Treatises about it.

*Lucas Pacioli*, a Franciscan Friar, commonly known by the Name of *Lucas de Burgo Sancti Sepulchri*, published at *Venice*, under the Title of, *A Compleat Treatise of Arithmetick and Geometry, Proportions and Equations*, the first Book at present extant on this Subject. It was printed so early as 1494, and is a very correct Treatise. He ascribes the Invention of Algebra to the *Arabians*, uses their Method, and treats very clearly of Quadratic Equations. After him, several Authors wrote on the same Subject in *Italy*, and in *Germany*; but still the Art advanced little 'till the famous *Jerom Cardan* printed, at *Nuremberg* in 1545, in Folio, a Treatise with this Title, *Artis magnæ, sive de Regulis Algebraicis Liber unus*; and soon after a smaller Piece, with the Title of *Sermo de Plus & Minus*, wherein were contained Rules for resolving Cubic Equations, which have since been called *Cardan's Rules*, though they were not invented by him, but, as himself owns, by *Scipio Ferreus* of *Bononia*, and *Tartalea*. The next celebrated Writer was a *French Monk*, whose Name was *Baeton*, better known to the Learned by his *Latin* Appellation of *Buteo*; he published in 1559 his *Logistica*, in which there was a Treatise of Algebra which gained him great Reputation: Yet his Excellency lay in a clear and copious Manner of writing, nor does it appear that he added any thing to what had been already discovered, except some Corrections as to *Tartalea's* Method of managing Cubic Equations.

Hitherto nothing was known in *Europe* of the *Greek* Analysis, but in 1575 *Xilander* published *Diophantus*, or at least a Part of his Works, which are still remaining; and this quickly changed the Face of Things, for it presently appeared that his was a nearer and more easy method, and withal opened a Path to much greater Discoveries, which was the Reason that succeeding Algebraists quitted the Terms made use of by *Arabic* Writers, and followed his. The Time in which *Diophantus* flourished

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is not thoroughly settled. *Vossius* thinks he lived in the second Century, but others place him in the fourth. His Works were known to the *Arabians*, and translated by them; nay, it is said, that they have still those seven Books of his Arithmetick, which are lost to us. The famous *Arabian* Historian *Abul Pharajus*, whose Works were published by the learned *Pocock*, not only mentions him, but ascribes to him the Invention of Algebra; but in this he is to be understood, as writing according to the Lights he had; for tho' it be true, that *Diophantus Alexandrinus* is the oldest Author we have which treats expressly of the *Analytic Art*, yet the Footsteps thereof are visible in much older Writers. *Theo*, who is thought to have explained the five first Propositions of the thirteenth Book of *Euclid* in the Analytic Way, gives the Honour of this Invention to *Plato*; and indeed, it seems very agreeable to his Genius, and Method of Reasoning on Mathematical Subjects. By the Junction of both Lights, and a proper Connection of the *Arabic* Method of Investigation with the *Greek* Terms, which were shorter and easier, Algebra quickly became a much more useful, as well as considerable Science, than it was before.

In our own Country, the first Writer upon Algebra that we know of was Dr. *Robert Record*, a Physician, who distinguished himself in the Reign of Queen *Mary*, by his Skill in the Mathematicks. He first published a Treatise of Arithmetick, which continued the Standard in that Branch of Knowledge for many Years, and in 1557 he sent abroad a second Part, under the Title of *Cos Ingenii, or the Whetstone of Wit*, which is a Treatise of Algebra; the Word *Cos* alluding to *Cossick* Numbers, or the Rule of *Cos*, by which Name, as we have before shewn, this Art was known abroad. This Treatise is really a great Curiosity, considering the Time in which it was published, and together with his other Works, must give us a high Idea of this Man's Industry and Application, whose Memory notwithstanding is almost buried in Oblivion. But, notwithstanding the early Publication  
of

of this Piece, and that some *English* Gentlemen had in their Travels acquired some Knowledge of this Kind, as appears by a *Spanish* Treatise of Algebra, published by *Pedro Nunnez* in 1567, yet it continued to be so little cultivated in *England*, that *John Dee*, in his Mathematical Preface prefixed to Sir *Henry Billingsley's* Translation of *Euclid*, printed at *London* in 1570, speaks of it in very high Terms, and as a Mystery scarce heard of by the Studious in the Mathematicks here. It is however plain, from some of his Annotations on *Euclid*, that he was tolerably versed therein, and was even acquainted with the Manner of applying it to Geometry. In 1579 *Leonard Digges*, a great Mathematician for those Times, printed a Treatise of Algebra in his *Stratoticos*; after which it came to be better known and more studied, to which contributed not a little, the Improvements made by the Author I am next to mention.

*Francis Viète*, better known by his *Latin* Name of *Franciscus Vieta*, was a Native of *Poitou*, in *France*, and Master of Requests to Queen *Margaret*, first Wife to King *Henry IV.* His Affection to the Mathematicks, and especially to this Part of it, was so strong, that he frequently passed three whole Days and Nights in his Study without eating, drinking, or sleeping, except a Nod now and then upon his Elbow\*. He, about the Year 1590, published a Treatise of Algebra in quite a new Method, and by a judicious Mixture of the *Greek* and *Arabian* Rules, with some Improvements of his own, introduced that Mode of Calculation which is still in Use, under the Title of *Specious Arithmetick*. Before his Time, only unknown Quantities were marked by Letters, but such as were known were set down in Figures according to the usual Notation: He made use of Letters for both, only with this Distinction, that the known Quantities he represented by Consonants, and the unknown by Vowels. By this Contrivance he greatly extended the Science, and which was more, shewed its Capacity of being farther extended. For, whereas former Algebraists had confined their

\* *Thuan.* Hist. A. D. 1603.

their Investigations to the particular Questions proposed to them, he by this Means produced Theorems capable of resolving all Demands of a like Nature, instead of particular Solutions. The learned Dr. *Wallis* has accounted very clearly for the new Title which *Vieta* gave to his Algebra. The *Romans* had a Method of stating Law Questions under general Names, such as *Titus* and *Sempronius*, *Caius* and *Mevius*, whence we derive our Way of using A, B, C, D, on such Occasions, which Method of stating the *Civilians* stile *Species*, in Opposition to the stating of *real Cases* by *true Names*. *Vieta* having made a Change of the same Nature in Algebra, and being, as we observed before, a Lawyer by Profession, he borrowed from that Science this Title of his new Invention, which was received with universal Applause. We have likewise many of his Works, under the Name of *Apollonius Gallus*, which he assumed on Account of his first attempting to restore the Works of *Apollonius Pergæus*. His Genius was so extensive, and his Penetration so great, that it enabled him to apply his Mathematical Knowledge to most Subjects; of which we have a particular Instance, in his decyphering the Letters which passed between the Court of *Spain*, and the Faction of the League in *France*, notwithstanding above five hundred different Characters were made use of in them. About the same Time flourished *Raphael Bombelli*, an *Italian*, who published at *Florence* a Treatise of Algebra, wherein he first taught how to reduce a biquadratic Equation to two Quadratics, by the Help of a Cubic.

Our own Countryman, Mr. *William Oughtred*, was the next great Improver of Algebra. Building, however, on what *Vieta* had already performed. He introduced such a Conciseness, and withal so plain and perspicuous a Method of investigating Geometrical Problems, as acquired him immortal Reputation. His *Clavis Mathematicæ*, or *Key of the Mathematicks*, was first published in 1631, and is perhaps the closest and most compendious System hitherto extant.

extant. In this Work he contented himself with the Solution of quadratic Equations, reserving those of higher Powers for another Work, which was his *Exegetis Numerosa*, which in later Editions is joined to his *Clavis*. In both Pieces there were abundance of Additions and Improvements, and the Doctrine of Proportions more fully and clearly stated than hitherto it had been; but the greatest Excellency in Mr. *Oughtred's* Book, was his Application of the Analytic Method to Geometry, which he did in a Variety of Cases, and enabled his Disciples to proceed still farther than himself had done. By Profession he was a Clergyman, and Rector of *Albury* in *Surry*, where he gave himself up entirely to his Studies, and to the Conversation of a very few Friends; he lived to the Age of Fourscore and Seven, and died then of Joy, on *May 1, 1660*, at hearing the House of Commons had voted the King's Return. Some have censured his *Clavis* as too short and obscure, and so indeed it might prove for such as were altogether unacquainted with these Studies, for whose Use it is plain enough he never designed it; but where Persons are acquainted with the Elements of Geometry and Algebra, and have that Sagacity and Attention which is necessary to make any considerable Progress in this Sort of Learning, Mr. *Oughtred's* Key will be still found a very useful Book, and its Style the most perfect in its Kind that has ever been used.

Contemporary with him was Mr. *Thomas Harriot*, an excellent Mathematician, and who made still greater Improvements in this Science. He is placed after *Oughtred*, tho' he died long before him, because his Book was not published till some Time after the first Edition of *Oughtred's Clavis*. It was then printed in a thin Folio by the Care of Mr. *Walter Warner*, under the Title of *Artis Analytica Praxis ad Aequationes Algebraicas novâ, expeditâ, & generali Methodo, resolvendas, Tractatus posthumus, &c. i. e.* A Treatise of the Analytic Art, containing a new, expeditious, and general Method of resolving Equations, a posthumous Tract, by the late learned Mr. *Thomas Harriot*.

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*riot.* The Publisher, Mr. *Warner*, prefixed a Preface of his own, containing a very judicious, tho' very concise, Representation of the several Parts of Algebra, their Nature and Dependance on each other, the Extent and Usefulness of this Art, and the Progress thereof to that Time. In Mr. *Harriot's* Book, Algebra takes a new Form, and from him alone it met with more Improvement than from all who had studied, or at least all who had written upon it, before him. He was indeed one of the greatest Men this Nation ever produced, and great Pity it was, that this Work of his did not appear in his Life-time, or that his other Pieces, which were of infinite Value, should be buried in Oblivion. The true Cause of the former seems to have been his Course of Life; he was a Dependant on the Earl of *Northumberland* and Sir *Walter Raleigh*, and afterwards upon Sir *Thomas Aylesbury*, to whom, if I am rightly informed, he left many of his Writings, and, as I hinted, the Reason of his not publishing them in his Life-time, seems to have been his Deference for his Benefactors. Happy had it been, if the rest of the Mathematical Works he left had been sent abroad (as in his Preface he seemed to promise they should) by the intelligent Editor of this excellent Work.

It is divided into two Parts; and the Author begins his Improvements by removing every thing that was useless, superfluous, or inelegant in former Methods; thus instead of Capitals, he introduced small Letters; instead of the Terms, Squares, Cubes, Surfsolids, &c. and their Contractions, he brought in the Powers themselves, which made the Operations much more easy, natural, and perspicuous than they were before. Having thus established a plain and accurate Notation, he proceeds to a Multitude of new Discoveries, of which, to the Number of twenty-three, the Reader may find a full, distinct, and very judicious Account, in the celebrated Treatise of Dr. *Wallis*. From this admirable Piece of Mr. *Harriot's*, *Des Cartes* took all the Improvements he pretended to make,



make, as the Doctor justly observes, and of which I shall furnish the Reader with some concise, and I think conclusive Proofs. *First*, It appears from all the Accounts we have of the Life of *Des Cartes*, that he was here in *England* when *Harriot's* Book was published, which being written in *Latin*, in a Branch of Learning about which that great Man was then very sedulous, it is easy to conceive that he was one of its most early Perusers; *Secondly*, It is certain that he did not publish any thing on this Subject before that Year; *Thirdly*, His Treatise of Geometry, wherein these new Improvements first appeared, was printed in *French* in 1637 without his Name, which in all Probability was to try what Opinion the World would have of them, and whether any of the *French* Mathematicians could discern whence they were taken; *Fourthly*, Though he suffered the two first Parts of his Book to be published in *Latin*, with his Name, in 1644; yet the third Part, relating to Geometry, did not appear till 1649, when it was published by *Francis Van Schooten*. These are probable Reasons only, but then, *Fifthly*, He follows *Harriot* distinctly in *Nineteen* several Discoveries; which that they should be made in the same Method and Manner, (except a few Mistakes) without consulting Mr. *Harriot*, is altogether incredible, and was so held to be even by his own Countrymen, when, thro' the Information of the Honourable Mr. *Cavendish*, they were made acquainted with Mr. *Harriot's* Book; *Sixthly*, There are some little Changes, particularly in the Marks made use of by *Des Cartes*, and which were never followed by any body, that plainly intimate he only introduced them, in order to disguise his Method; *Seventhly*, It appears that *Des Cartes* himself was acquainted with the Charge brought against him upon this Head, and yet he never thought fit to justify himself, nor did ever so much as declare that he had not seen the Book he was said to have copied. On the whole therefore, there is all the Reason in the World to believe, that the Honour due to the great Improvement of this Science,

which fitted it for all that it has received since, from Foreigners or *Englishmen*, belongs to our Author *Harriot*, and not to *Des Cartes*, who only accommodated these Discoveries to Geometrical Subjects.

After him Dr. *John Pell*, who was Resident for the *Commonwealth of England in Switzerland*, published some new Discoveries. The Method he took of doing it was this, he recommended to Mr. *Thomas Brancker* a Treatise of Algebra written in the *German Language* by *Rhoni*, which when he had translated, the Doctor revised, altered and added to it. In this Piece there are a great many curious Things relating especially to *Diophantine Algebra*, but delivered very obscurely, insomuch, that the learned Dr. *Wallis* seems to be in doubt, whether himself had reached Dr. *Pell's* true Meaning. Yet, to this Gentleman, who wrote in so perplexed a Way, we stand indebted for the Invention of the Register; a Method of great Use, especially to Beginners, the Practice of which was what chiefly recommended *Kersey's Algebra*, and which is constantly and judiciously preserved throughout the following Treatise. It is very likely, that the Darkeness complained of in Dr. *Pell's* Writings might be owing to his Circumstances as well as Temper, for he was a very bad Oeconomist, not through any Vice or Extravagancy, but by a Neglect of his private Affairs, and spending all his Time in Study.

As for the Rules of *John Van Hudde*, Mr. *Merry*, *Erasmus Bartholine*, Mr. *Hugens*, and others, I do not take Notice of them, because in reality they are no more than Improvements on, or Deductions from, *Harriot*. The same Thing may be said of what has been written by Mess. *Farmat*, *de Billy*, *Fernicle*, and other *French Mathematicians*, who only proposed Problems for other People to resolve, and reserved their own Methods of Solution as impenetrable Secrets: A Practice, which, however it might intitle them to the Admiration of the Age in which they lived, can give them no just Claim to the Praise of Posterity; since if we reap any Benefit from their Discoveries,

veries, it is indirectly, and in a Manner against their Intentions.

Dr. *Wallis* himself has also made some very considerable Improvements in this Science, especially in respect to impossible Roots in superior Equations; and what he left unperfected has been supplied by the ingenious Mr. *Abraham De Moivre*, whose accurate Performance on that Subject has been lately published, in the Algebra of Dr. *Saunderson*.

In 1655 Dr. *Wallis* published his *Arithmetica Infinitorum*, in which he squared a Series of Curves, and shewed that if this Series could be interpolated in the middle Spaces, the Interpolation would give the Quadrature of the Circle. This Treatise fell into the Hands of the ingenious Sir *Isaac* then Mr. *Newton*, in the Year 1664, when that Gentleman was about Two and Twenty; and he by a Sagacity peculiar to himself, and which can never be enough admired, derived from this Hint his celebrated Method of Infinite or Converging Series. In 1665, he computed the Area of the Hyperbola by this Series to Fifty-two Figures, which having communicated to Dr. *Barrow*, he prevented Mr. *Nicholas Mercator's* running away with the Reputation of this Discovery, who in 1668 published the Quadrature of the Hyperbola by an infinite Series. This was received with universal Applause, and yet Mr. *Newton* far exceeded him; since, without stopping at the Hyperbola, he extended this Method by general Forms to all Sorts of Curves, even such as are Mechanical, to their Quadratures, Rectifications, and Centers of Gravity, to the Solids formed by their Rotations, and to the Superficies of those Solids; so that supposing their Determinations to be possible, this Series stopped at a certain Point, or at least their Sums were given by stated Rules. But if the absolute Determinations were impossible, they could yet be infinitely approximated, as he likewise shewed, and which, as a *French* Writer justly observes, is the happiest and most refined Contrivance for supplying the Defects of human Knowledge, that Man's Imagination could possibly invent. It is also certain, that he attained his

Invention of Fluxions by that Time he was Four and Twenty, but his Modesty was so great, that he forbore to publish his Discovery, which was the sole Reason that the Honour of it was ever disputed with him.

In 1707, he first published a System of Algebra under the Title of *Universal Arithmetick*, and in 1722 gave another Edition of it, wherein are contained all his Improvements in that Art.

From the Rules by him laid down, still farther Lights were struck out by succeeding Mathematicians, such as Dr. *Edmund Halley*, who published in the *Philosophical Transactions*, a Method of finding the Roots of Equations without any previous Reduction, and the Construction of Equations of the 3d and 4th Power, by the Help of a Circle and Parabola. Mr. *J. Colson*, who obliged the learned World with a universal Resolution, Geometrical and Mechanical, of Cubic and Biquadratic Equations. Mr. *Colin Mac Laurin*, in his Treatise of impossible Roots, and many others too long to be enumerated here.

But after all these Discoveries and Improvements, there has still been a general Complaint, that hitherto we have had no Book of Algebra plain enough to instruct such as are inclined to study this Science without farther Assistance, or who live in Places where it is not to be had. To obviate this Objection, the following Treatise was drawn up, which will be found to contain a clear and copious System of Algebra, delivered in so easy and natural a Method, and with such Perspicuity and Condescension to the Feebleness of the Understanding, when first applied to this kind of Study, that I felicitate myself on having prevailed upon its Author to make it publick, as I am persuaded it will be of general Use, in preventing young People from being discouraged at their first Entrance into Algebra, which has hitherto hindered Numbers from cultivating their Inclinations to the Mathematicks.

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# A L G E B R A.

**H**AVING given the Reader an Historical Account of this Science in the Introduction, we are now to explain the Signs and Characters used by Analytic Writers, and mention those Axioms or Self-evident Principles of Truth and Certainty, which are the Foundations of this celebrated Science.

<i>Signs.</i>	<i>Names.</i>	<i>Significations.</i>
$+$ }	$\{$ <i>Plus or more.</i>	$\left\{ \begin{array}{l} \text{The Sign of } \textit{Addition}; \text{ as } 8 + 4, \text{ is } 8 \text{ is} \\ \text{to be added to } 4, \text{ and } m + n \text{ signifies the} \\ \text{Number represented by } m, \text{ is to be added} \\ \text{to the Number represented by } n; \text{ again,} \\ 2 + 3 + 5, \text{ signifies they are all to be added} \\ \text{into one Sum, and } b + m + d \text{ signifies that} \\ \text{the Numbers represented by } b, m, \text{ and } d \\ \text{are to be added into one Sum.} \end{array} \right.$
$-$ }	$\{$ <i>Minus or less.</i>	$\left\{ \begin{array}{l} \text{The Sign of } \textit{Subtraction}; \text{ as } 5 - 2, \text{ is} \\ 5 \text{ less by } 2, \text{ or } 2 \text{ is to be subtracted from} \\ 5, \text{ and } a - b \text{ is } a \text{ less } b, \text{ or the Number} \\ \text{represented by } b \text{ is to be subtracted from} \\ \text{the Number represented by } a; \text{ and } 9 - 2 \\ - 3, \text{ is that from } 9 \text{ there is to be sub-} \\ \text{tracted } 2, \text{ and from the Remainder } 3 \text{ is} \\ \text{to be subtracted.} \end{array} \right.$
$\times$ }	$\{$ <i>Into or with.</i>	$\left\{ \begin{array}{l} \text{The Sign of } \textit{Multiplication}; \text{ as } 5 \times 7, \text{ is} \\ 5 \text{ is to be multiplied by } 7, \text{ and } a \times b, \text{ is} \\ \text{the Number represented by } a, \text{ is to be} \\ \text{multiplied by the Number represented by} \\ b; \text{ and } 7 \times 3 \times 2, \text{ is that } 7, 3, \text{ and } 2, \text{ are} \\ \text{to be multiplied together, which Product} \\ \text{is } 42. \end{array} \right.$

$\div \{ \}$  By.

The Sign of *Division*; as  $8 \div 4$ , that is 8 is to be divided by 4, and  $x \div y$ , that is, the Number represented by  $x$ , is to be divided by the Number represented by  $y$ ; or sometimes they are placed like Vulgar

Fractions thus  $\frac{8}{4}$ , that is, 8 is to be di-

vided by 4, and  $\frac{x}{y}$ , that is, the Number represented by  $x$ , is to be divided by the Number represented by  $y$ .

$= \{ \}$  Equal.

The Sign of *Equality* or *Equation*; thus  $9 = 9$ , that is, 9 is equal to 9, and  $2 + 3 = 5$ , that is, 2 added to 3, is equal to 5: Again,  $m = n + y$ , that is, the Number represented by  $m$  is equal to the Number represented by  $n$ , added to the Number represented by  $y$ ; and  $y - x = a + b$ ; that is, the Number represented by  $y$  being lessened by the Number represented by  $x$ , the Remainder is equal to the Number represented by  $a$ , added to the Number represented by  $b$ .

$:: \{ \}$  So is.

The Sign of *Proportion*, or what is commonly called the Rule of Three, and is placed between the two middle Number thus,  $3 : 5 :: 6 : 10$ , that is, as 3 is to 5, so is 6 to 10; and  $a : b :: c : d$ , that is, as the Number represented by  $a$ , is to the Number represented by  $b$ , so is the Number represented by  $c$  to the Number represented by  $d$ .

$\odot \{ \}$  Involution.

The Sign of *Involution*, or raising any Number or Quantity to the *Square*, *Cube*, or any other Power; and the Height of the Involution is generally expressed by the Number after the Sign thus,  $7 \odot 2$ , is 7 is to be involved to the Square or second Power; and  $7 \odot 3$ , is 7 is to be involved or raised to the Cube or third Power; and  $a \odot 2$ , is  $a$  is to be involved to the Square or second Power.

The

$w \} \} \text{Evolution.}$

The Sign of *Evolution*, or the extracting of Roots; and the Root that is taken is likewise expressed by the Figure that follows the Sign, thus  $9w2$ , is the Square Root of 9 is to be extracted, and  $27w3$ , is the Cube Root of 27 is to be extracted, and  $aa w 2$ , is the Square Root of  $aa$  is to be extracted.

$\checkmark \} \} \text{Irrationality, or a Surd Root.}$

The Sign of *Irrationality*, or of a Surd Root; that is, the Number or Quantity has not such a Root as is required to be extracted; thus the Square Root of 2 will be expressed thus  $\sqrt{2}$ , and the Square Root of 5 thus  $\sqrt{5}$ , and the Cube Root of 4 thus  $\sqrt[3]{4}$ , the little Figure standing over the Sign being 3, shews it to be the Cube Root; again,  $\sqrt[3]{15}$  is the Cube Root of 15, and where there is no such Figure over the Sign, it signifies the Square Root only.

Now before we go farther, it will be necessary to inform the Reader, that where any Number is joined to a Quantity, it shews how many Times that Quantity is taken; thus,  $4a$  is four times  $a$ , or the Number represented by  $a$  is to be taken four times; and  $7m$  is seven times  $m$ , and if  $y$  was to be taken seven times, it may be expressed thus  $7y$ .

These Numbers are called *Co-efficients*, or *Fellow-Factors*, as they multiply the Quantity; and if any Quantity is without a Co-efficient, then it is always implied that *Unity*, or 1, is the Co-efficient of that Quantity; thus  $a$  is the same as  $1a$ , and  $y$  the same as  $1y$ ; for when the Co-efficient is only *Unity*, or 1, it is generally omitted.

Quantities that are expressed or represented by single Letters, or several joined together like a Word, as  $a$ ,  $b$ ,  $ab$ ,  $anx$ ,  $7yz$ , are called simple or single Quantities.

But when these are connected by the Signs  $+$  or  $-$ , as  $a+b$ ,  $am-a$ ,  $dn+az$ , they are called compound Quantities.

And sometimes Quantities are set down in the Manner of

Vulgar Fractions, thus,  $\frac{a}{b}$ ,  $\frac{a+b}{n}$ ,  $\frac{n}{x+y}$ .

The Sign that connects the Quantities belongs to that which follows the Sign, thus,  $a + b$ , where the Sign  $+$  belongs to the Quantity  $b$ ; again,  $a - c + d$ , the Sign  $-$  belongs to the Quantity  $c$ , and the Sign  $+$  to the Quantity  $d$ .

As to those single Quantities which have no Sign before them, it is always understood they have the Sign  $+$ ; thus  $a$  is the same as  $+a$ , and  $m$  is the same as  $+m$ ; and therefore if single Quantities are to have the Sign  $+$ , it is commonly omitted, as they are usually set down without any Sign; but the Sign  $-$  is never omitted, but always placed before the Quantity to which it belongs.

And in Compound Quantities, if the first or leading Quantity has no Sign, then it is always understood to have the Sign  $+$ , thus,  $a + b$  is the same as  $+a + b$ , and  $a - b$  is the same as  $+a - b$ ; therefore in Compound Quantities, if the first or leading Quantity is to have the Sign  $+$ , it is generally omitted; but in these Compound Quantities, as well as in Simple Quantities, the Sign  $-$  is never omitted, but always placed before the Quantity to which it belongs.

Letters set or joined together like a Word signifies the Product or Rectangle of these Letters, thus,  $ab$  is the Product of  $a$  multiplied by  $b$ , and  $dny$  is the Product of  $d$ ,  $n$ , and  $y$ , multiplied together.

The Operations in *Algebra* are founded on these *Axioms*.

#### A X I O M 1.

If equal Quantities are added to equal Quantities, the Sum of these Quantities will be equal.

#### A X I O M 2.

If equal Quantities are taken or subtracted from equal Quantities, the Quantities remaining will be equal.

#### A X I O M 3.

If equal Quantities are multiplied by equal Quantities, their Products will be equal.

#### A X I O M

## A X I O M 4.

If equal Quantities are divided by equal Quantities, their Quotients will be equal.

## A X I O M 5.

If there are several Quantities that are equal to one and the same Thing, those Quantities are equal one to another.

The Reader having premised these Things, and understanding what the Signs are intended to express, he may proceed to the Rules of the Science; and if at first he meets with some little Difficulties about the Signs and Co-efficients, I would recommend him to read the foregoing Pages again; and if that and another Essay or two does not remove the Difficulties of any particular Example, then to omit that and proceed to the next, in which perhaps he may succeed, and that may cause the Difficulty in the other to vanish.

# A D D I T I O N,

*In which there are three Cases.*

(1.) *Case 1.* **W**HEN the Quantities are alike, and their Signs are both affirmative, or both negative, add the Co-efficients or prefixt Numbers together, and to their Sum join the Quantities, prefixing to them the Sign they have in the *Example*.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
To	2 <i>a</i>	5 <i>m</i>	— 4 <i>y</i>	— 2 <i>z</i>
Add	<u>3 <i>a</i></u>	<u>2 <i>m</i></u>	<u>— 3 <i>y</i></u>	<u>— 6 <i>z</i></u>
Sum	5 <i>a</i>	7 <i>m</i>	— 7 <i>y</i>	— 8 <i>z</i>

*Exam. 1.* The Co-efficients are 2 and 3, which added together make 5, to which joining *a* the Quantity, it is 5 *a*, and no Sign

Sign being prefix to either  $2a$  or  $3a$ , the *affirmative* Sign is understood as prefix to both; hence  $5a$ , or  $-5a$  is the Sum required.

*Exam. 2.* The Co-efficients are 5 and 2, which being added make 7, to which joining  $m$ , it is  $7m$ , the Sum required; for the Signs of  $5m$  and  $2m$  are both *affirmative*, by what was said in the last Example.

*Exam. 3.* The Co-efficients are 4 and 3, which being added make 7, to which joining  $y$  it becomes  $7y$ ; but as  $4y$  and  $3y$  have both the Sign — before them, therefore prefix the Sign — to  $7y$ , and then —  $7y$  is the Sum required.

*Exam. 4.* The Co-efficients are 2 and 6, which added make 8, to which joining  $z$ , it becomes  $8z$ , and prefixing the Sign — for the Reason in the last Example, we have —  $8z$ , the Sum required.

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
To	$15my$	$-14azx$	$4ady$	$-16ymd$
Add	$7my$	$-2azx$	$3ady$	$-12ymd$
Sum	$22my$	$-16azx$	$7ady$	$-28ymd$

*Exam. 5.* The Sum of the Co-efficients 15 and 7 is 22, to which joining  $my$ , it is  $22my$ , the Sum required; for  $15my$  and  $7my$  have both the *affirmative* Sign, there being no Sign prefix.

*Exam. 6.* The Sum of the Co-efficients 14 and 2 is 16, to which joining  $azx$ , it is  $16azx$ , to which prefixing the Sign —, as both the Quantities to be added have that Sign, then is —  $16azx$  the Sum required.

*Exam. 7.* The Sum of the Co-efficients 4 and 3 is 7, to which joining  $ady$ , it is  $7ady$ , and both the Quantities having the *affirmative* Sign, therefore  $7ady$  is the Sum required.

*Exam. 8.* The Sum of the Co-efficients 16 and 12 is 28, to which joining  $ymd$ , it is  $28ymd$ , to which prefixing the Sign —, as both the Quantities to be added have that Sign, then is —  $28ymd$  the Sum required.

	<i>Exam. 9.</i>	<i>Exam. 10.</i>	<i>Exam. 11.</i>	<i>Exam. 12.</i>
To	$2m y$	$1a n$	$21d y$	$-d a$
Add	$3m y$	$2a n$	$d y$	$-d a$
Sum	$5m y$	$3a n$	$22d y$	$-2d a$

*Exam.*

*Exam. 11.* The Co-efficient are 21 and 1, for there being no Co-efficient prefixt to  $dy$ , *Unity*, or 1, is always understood in such Cases to be the Co-efficient; hence the Sum is  $22 dy$ .

*Exam. 12.* There being no Co-efficient prefixt to either of the Quantities, *Unity*, or 1, is the Co-efficient to each; and 1 being added to 1 makes 2, to which joining  $da$ , it is  $2da$ , to which prefixing the *negative* Sign, we have  $-2da$ , the Sum required.

(2.) If there are two or more Quantities connected by the Signs  $+$  or  $-$ , and are alike to two or more Quantities connected by the Signs  $+$  or  $-$ , they are added as in the former Examples, only taking due Care that the Quantities which compose their Sum are connected with their proper Signs, according to the Rule, as in the following Examples.

	<i>Exam. 13.</i>	<i>Exam. 14.</i>	<i>Exam. 15.</i>
To	$2a + 7b$	$6ma + 5y$	$21ma + 2yd$
Add	$3a + 2b$	$2ma + 3y$	$3ma + 3yd$
Sum	$5a + 9b$	$8ma + 8y$	$24ma + 5yd$

*Exam. 13.* Is  $2a + 7b$  to be added to  $3a + 2b$ . The Quantities being disposed as in the Example, it follows from former Examples that  $2a$  being added to  $3a$  makes  $5a$ , and  $7b$  added to  $2b$  makes  $9b$ ; but as  $7b$  and  $2b$  have both the affirmative Sign, to  $5a$  connect  $9b$  with the Sign  $+$ ; hence  $5a + 9b$  is the Sum required.

*Exam. 14.* Is  $6ma + 5y$  to be added to  $2ma + 3y$ . Now by the former Examples  $6ma$  being added to  $2ma$  is  $8ma$ , and  $5y$  being added to  $3y$  is  $8y$ ; but as  $5y$  and  $3y$  have both the affirmative Sign, to  $8ma$  connect  $8y$  with the Sign  $+$ ; so will  $8ma + 8y$  be the Sum required.

*Exam. 15.* Is  $21ma + 2yd$  to be added to  $3ma + 3yd$ . Now by the former Examples  $21ma$  being added to  $3ma$ , the Sum is  $24ma$ ; and  $2yd$  being added to  $3yd$ , the Sum is  $5yd$ . But as  $2yd$  and  $3yd$  have both the affirmative Sign, therefore connecting  $24ma$  and  $5yd$  with the Sign  $+$ , we have  $24ma + 5yd$ , the Sum required.

	<i>Exam. 16.</i>	<i>Exam. 17.</i>	<i>Exam. 18.</i>
To	$-7da - 15m$	$9ma - 14nd$	$-2mn + 15yd$
Add	$-2da - 4m$	$3ma - 3nd$	$-4mn + 4yd$
Sum	$-9da - 19m$	$12ma - 17nd$	$-6mn + 19yd$

*Exam.*

*Exam. 16.* Is  $-7da - 15m$  to be added to  $-2da - 4m$ . Now  $7da$  added to  $2da$  is  $9da$ ; but as both these Quantities have the Sign  $-$ , prefix the *negative* Sign to  $9da$ , and then it is  $-9da$ . Again,  $15m$  added to  $4m$  is  $19m$ ; and both these Quantities having likewise the *negative* Sign, prefix it to  $19m$ ; whence the Sum required is  $-9da - 19m$ .

*Exam. 17.* Is  $9ma - 14nd$  to be added to  $3ma - 3nd$ . Now  $9ma$  added to  $3ma$ , is  $12ma$ ; and both these Quantities having the Sign  $+$ , place down  $12ma$  as in the Example: Then  $14nd$  added to  $3nd$ , is  $17nd$ ; but both these Quantities having the Sign  $-$ , place the Sign  $-$  before  $17nd$ , and the Sum required is  $12ma - 17nd$ .

*Exam. 18.* Is  $-2mn + 15yd$  to be added to  $-4mn + 4yd$ . Now  $2mn$  added to  $4mn$ , is  $6mn$ ; but both these Quantities having the *negative* Sign, prefix the Sign  $-$  to  $6mn$ , and then it is  $-6mn$ . And  $15yd$  added to  $4yd$ , is  $19yd$ ; and both these Quantities having the *affirmative* Sign, prefix the Sign  $+$  to  $19yd$ ; hence the Sum is  $-6mn + 19yd$ .

	<i>Exam. 19.</i>	<i>Exam. 20.</i>	<i>Exam. 21.</i>
To	$9yd - 7a$	$14yd + 15a$	$-14y + d$
Add	$2yd - a$	$2yd + a$	$-y + d$
Sum	$11yd - 8a$	$16yd + 16a$	$-15y + 2d$

*Exam. 19.* When you come to add  $-7a$  to  $-a$ , there being no Co-efficient prefixt to  $a$ , Unity, or 1, is always in such Cases the Co-efficient; and then by what has been already taught,  $-7a$  being added to  $-a$ , the Sum is  $-8a$ , as in the Example.

*Exam. 20.* And when  $15a$ , is to be added to  $a$ , the Sum is for the same Reason  $16a$ .

*Exam. 21.* And  $-14y$  being added to  $-y$ , the Sum is  $-15y$ , and  $d$  being added to  $d$ , for the same Reason the Sum is  $2d$ , or  $+2d$ .

(3.) *Case 2.* When the Quantities are alike, but the Signs are one *affirmative*, and the other *negative*, subtract the lesser Co-efficient from the greater, to the Remainder join the Quantity, and prefix to it the Sign of the greatest Co-efficient.

It is of no Signification whether the Quantity that has the greatest Co-efficient stands above or below.



<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
To     5 <i>a</i>	16 <i>m</i>	21 <i>a d</i>	14 <i>m z</i>
Add — 2 <i>a</i>	— 12 <i>m</i>	— 7 <i>a d</i>	— 5 <i>m z</i>
Sum     3 <i>a</i>	4 <i>m</i>	14 <i>a d</i>	9 <i>m z</i>

*Exam. 1.* The Co-efficient 2 subtracted from 5 leaves 3, to which joining *a* it is 3 *a*, but the Sign of 5 the greatest Co-efficient is *affirmative*, therefore 3 *a* or + 3 *a* is the Sum required.

*Exam. 2.* The Co-efficient 12 subtracted from 16 leaves 4, to which joining *m* it is 4 *m*, but the Sign of 16 the greatest Co-efficient is *affirmative*, therefore 4 *m* or + 4 *m* is the Sum required.

*Exam. 3.* The Co-efficient 7 subtracted from 21 leaves 14, to which joining *a d* it is 14 *a d*, but the Sign of 21 the greatest Co-efficient is *affirmative*, hence 14 *a d* or + 14 *a d* is the Sum required.

*Exam. 4.* The Co-efficient 5 subtracted from 14 leaves 9, to which joining *m z* it is 9 *m z*, but the Sign of 14 the greatest Co-efficient is *affirmative*, hence 9 *m z* or + 9 *m z* is the Sum required.

<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
To   — 14 <i>m</i>	— 9 <i>y</i>	5 <i>z</i>	9 <i>a m</i>
Add     7 <i>m</i>	2 <i>y</i>	— 9 <i>z</i>	— 14 <i>a m</i>
Sum   — 7 <i>m</i>	— 7 <i>y</i>	— 4 <i>z</i>	— 5 <i>a m</i>

*Exam. 5.* The Co-efficient 7 subtracted from 14 leaves 7, to which joining *m* it is 7 *m*, but the Sign of 14 the greatest Co-efficient being —, prefix that Sign to 7 *m*, then is — 7 *m* the Sum required.

*Exam. 6.* The Co-efficient 2 subtracted from 9, there remains 7, to which joining *y* it is 7 *y*, but the Sign of 9 the greatest Co-efficient being —, prefix that Sign to 7 *y*, and we have — 7 *y*, the Sum required.

*Exam. 7.* The Co-efficient 5 subtracted from 9 leaves 4, to which joining *z* it is 4 *z*, but the Sign of 9 the greatest Co-efficient being negative, prefix the Sign — to 4 *z*, and we have — 4 *z*, the Sum required.

*Exam. 8.* The Co-efficient 9 subtracted from 14 leaves 5, to which joining  $am$  it is  $5am$ , but the Sign of 14 the greatest Co-efficient being negative, prefix the Sign  $-$  to  $5am$ , and we have  $-5am$ , the Sum required.

	<i>Exam. 9.</i>	<i>Exam. 10.</i>	<i>Exam. 11.</i>	<i>Exam. 12.</i>
To	$7am$	$-8ad$	$-14ym$	$-ay$
Add	$-am$	$9ad$	$16ym$	$7ay$
Sum	$6am$	$ad$	$2ym$	$6ay$

*Exam. 9.* The Co-efficient of  $-am$  being Unity, or 1, which subtracted from 7 leaves 6, to which joining  $am$  it is  $6am$ , prefixing to it the Sign of 7, the greatest Co-efficient, we have  $6am$  or  $+6am$  the Sum required.

*Exam. 10.* The Co-efficient 8 subtracted from 9 leaves 1, to which joining  $ad$  we have  $1ad$  or  $ad$ , which having already the Sign of 9, the greatest Co-efficient, hence  $ad$  is the Sum required.

*Exam. 12.* The Co-efficient of  $-ay$  being Unity, or 1, which subtracted from 7 leaves 6, to which joining  $ay$  it is  $6ay$ , which having the same Sign with 7, the greatest Co-efficient,  $6ay$  is the Sum required.

4. And if there are several Quantities connected by the different Signs of  $+$  and  $-$ , to be added to several Quantities connected by the different Signs of  $+$  and  $-$ , the Quantities being alike, are added as in the second Article, only taking Care to prefix the Signs, according to the Directions in the first and third Articles.

	<i>Exam. 13.</i>	<i>Exam. 14.</i>	<i>Exam. 15.</i>
To	$14a + 7m$	$-15my - 14az$	$17ay + 8am$
Add	$-8a - 3m$	$7my + 12az$	$-3ay - 5am$
Sum	$6a + 4m$	$-8my - 2az$	$14ay + 3am$

*Exam. 13.* Is  $14a + 7m$  to be added to  $-8a - 3m$ . Now by the Rule at Art. 3. the Difference between the Co-efficients 14 and 8 is 6, to which joining  $a$  it is  $6a$ , but 14 the greatest Co-efficient having the affirmative Sign, hence  $6a$  is the Sum of  $14a$  added to  $-8a$ . And the Difference between 7 and 3 the Co-efficients of  $m$  being 4, to which joining  $m$  it is  $4m$ , but as 7 the greatest Co-efficient has the affirmative Sign, therefore to  $6a$  connect  $4m$  with the Sign  $+$ , so is  $6a + 4m$  the Sum required.

*Exam.*

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*Exam. 14.* Where  $-15\ my - 14\ az$  is to be added to  $7\ my + 12\ az$ . Now the Difference between 15 and 7 the two Co-efficients of  $my$  is 8, to which joining  $my$  it is  $8\ my$ , but as 15 the greatest Co-efficient hath the *negative* Sign, therefore prefix the Sign  $-$  to  $8\ my$ , and it is  $-8\ my$ : And the Difference between 14 and 12 the two Co-efficients of  $az$  being 2, to which joining  $az$  it is  $2\ az$ , but as 14 the greatest Co-efficient has the *negative* Sign, therefore to  $-8\ my$  connect  $2\ az$  with the Sign  $-$ , so is  $-8\ my - 2\ az$  the Sum required.

*Exam. 15.* The Difference between 17 and 3 the two Co-efficients of  $ay$  is 14, to which joining  $ay$  it is  $14\ ay$ , but as 17 the greatest Co-efficient has the *affirmative* Sign, therefore place down  $14\ ay$  or  $+14\ ay$ . And the Difference between 8 and 5 the two Co-efficients of  $am$  is 3, to which joining  $am$  it is  $3\ am$ , but as 8 the greatest Co-efficient has the *affirmative* Sign, therefore prefix the Sign  $+$  to  $3\ am$ , so is  $14\ ay + 3\ am$  the Sum required.

*Exam. 16.*

*Exam. 17.*

*Exam. 18.*

$$\begin{array}{r} \text{To } -7\ a + 16\ m \\ \text{Add } \quad 3\ a - \quad 4\ m \\ \hline \text{Sum } -4\ a + 12\ m \end{array}$$

$$\begin{array}{r} -15\ y + \quad 7\ p \\ \quad 7\ y - 11\ p \\ \hline -8\ y - \quad 4\ p \end{array}$$

$$\begin{array}{r} 7\ am - 16\ y \\ -11\ am + 18\ y \\ \hline -4\ am - 2\ y \end{array}$$

*Exam. 16.* By Art. 3. the Difference between 7 and 3 the two Co-efficients of  $a$  is 4, to which joining  $a$  it is  $4\ a$ , but as 7 the greatest Co-efficient has the *negative* Sign, therefore prefix the Sign  $-$  to  $4\ a$ , and it is  $-4\ a$ . And the Difference between 16 and 4 the two Co-efficients of  $m$  is 12, to which joining  $m$  it is  $12\ m$ , but 16 the greatest Co-efficient having the *affirmative* Sign, prefix the Sign  $+$  to  $12\ m$ , so is  $-4\ a + 12\ m$  the Sum required.

*Exam. 17.* By Art. 3. the Difference between 15 and 7 is 8, to which joining  $y$  it is  $8\ y$ , but 15 the greatest Co-efficient having the *negative* Sign, prefix the Sign  $-$  to  $8\ y$ , and it is  $-8\ y$ . And the Difference between 7 and 11 the two Co-efficients of  $p$  is 4, to which joining  $p$  it is  $4\ p$ , but as 11 the greatest Co-efficient has the *negative* Sign, therefore prefix the Sign  $-$  to  $4\ p$ , and it is  $-8\ y - 4\ p$  the Sum required.

*Exam. 18.* By Art. 3. the Difference between 7 and 11 is 4, to which joining  $am$  it is  $4\ am$ , but as 11 the greatest Co-efficient has the *negative* Sign, therefore prefix the Sign  $-$  to  $4\ am$ , and it is  $-4\ am$ . And the Difference between 16

and 18 is 2, to which joining  $y$  it is  $2y$ , but as 18 the greatest Co-efficient has the *affirmative* Sign, therefore prefix the Sign  $+$  to  $2y$ , so is  $-4am + 2y$  the Sum required.

Exam. 19.

Exam. 20.

Exam. 21.

To	$14my - ma$	$-5yd + 15z$	$-14dy + 5mp$
Add	$-3my + 4ma$	$yd - 3z$	$dy - mp$
Sum	$11my + 3ma$	$-4yd + 12z$	$-13dy + 4mp$

Exam. 19. The Co-efficient of  $-ma$  is 1, which being by Art. 3. subtracted from 4 leaves 3, to which joining  $ma$  it is  $3ma$ , as in the Answer, and by the same Method in

Exam. 20. If  $-5yd$  is added to  $yd$  or  $1yd$ , the Sum is  $-4yd$ ; and likewise in

Exam. 21. If  $-14dy$  is added to  $dy$  or  $1dy$ , the Sum is  $-13dy$ .

5. If the Quantities are alike and the Co-efficients are equal, but the Signs are one *affirmative*, and the other *negative*, these being added together destroy each other, or the Sum of them is a *Cypher* or *nothing*.

Exam. 1.

Exam. 2.

Exam. 3.

Exam. 4

To	$7a$	$-5y$	$14m$	$5ya$
Add	$-7a$	$5y$	$-14m$	$-5ya$
Sum	0	0	0	0

Exam. 1. By Art. 3. the Signs being unlike the Co-efficients are to be subtracted, but 7 taken from 7 leaves 0, and if to this we join  $a$  it is  $0a$ , or *no times a*, that is, the Quantity  $a$  is to be taken no times or not at all, which is the same as *nothing*: So in the fourth Example, if 5 is subtracted from 5, there remains 0, or *nothing*, to which if we join  $ya$ , we then have *no times ya*, or *nothing*.

(6.) *Case 3.* When the Quantities are unlike, that is, the Letters are different, then set them down one after the other, with the same Co-efficients and Signs they have in the Example, and this is the Sum required.

And they may be set in any Order, that is, any Quantity may be set first, in the middle or last, it being not material how they are ranged, so as they are but connected with their proper Signs.

Exam.

# A D D I T I O N.

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	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
To	$2a$	$3m$	$a + d$
Add	$3d$	$5a$	$2y$
Sum	$2a + 3d$	$3m + 5a$	$a + d + 2y$

*Exam. 1.* The Quantities or Letters being unlike, I place down  $2a$ , and because  $3d$  has the Sign  $+$ , therefore after the  $2a$  put  $+ 3d$ , so is  $2a + 3d$  the Sum required.

*Exam. 2.* Having put down the  $3m$ , after that put  $+ 5a$  the other Quantity with its Sign, so is  $3m + 5a$  the Sum required.

*Exam. 3.* Having put down  $a$ , after that put  $+ d$ , and after that  $+ 2y$ , so is  $a + d + 2y$  the Sum required.

	<i>Exam. 4.</i>	<i>Exam. 5.</i>
To	$2a - 7m$	$2a + 15$
Add	$3y + 5z$	$z - 7d$
Sum	$2a - 7m + 3y + 5z$	$2a + 15 + z - 7d$

*Exam. 4.* Begin and place down  $2a$ , after that  $- 7m$ , after that  $+ 3y$ , and after that  $+ 5z$ , so is  $2a - 7m + 3y + 5z$  the Sum required.

*Exam. 5.* Begin and place down  $2a$ , after that  $+ 15$ , after that  $+ z$ , and after that  $- 7d$ , so is  $2a + 15 + z - 7d$  the Sum required.

To	$7m + 15y$	$- 15m + 7a$
Add	$- 4a + mn$	$8y - 2b$
Sum	$7m + 15y - 4a + mn$	$- 15m + 7a + 8y - 2b$

To	$16 + 7m$	$- 14m - 15y$
Add	$- 2a - 8d$	$a - 7$
Sum	$16 + 7m - 2a - 8d$	$- 14m - 15y + a - 7$

Examples wherein all the foregoing Cases are promiscuously used.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>
To	$7a - 15d + m$	$- 8a + 7m - 21x$
Add	$5a + 18d$	$11a - 12m + 5y$
Sum	$12a + 3d + m$	$3a - 5m - 21x + 5y$

*Exam. 1.*  $7a$  added to  $5a$  makes  $12a$ , by Art. 1. and  $- 15d$  added to  $18d$  makes  $3d$ , by Art. 3. and there being no Quantity like  $m$ , that must be placed by itself, by Art. 6. and connect-  
ing

ing these Quantities with their proper Signs we have  $12a + 3d + m$ , the Sum required.

*Exam. 2.* —  $8a$  added to  $11a$  makes  $3a$ , by Art. 3. and  $7m$  added to  $-12m$  is  $-5m$ , by the same, but  $21x$  and  $5y$  being different, place them down one after another as at Art. 6. so is  $3a - 5m - 21x + 5y$  the Sum required.

*Exam. 3.*

$$\begin{array}{r} \text{To} \quad -15a + 14m - 16 \\ \text{Add} \quad 7a - 14m + y \\ \hline \text{Sum} \quad -8a - 16 + y \end{array}$$

*Exam. 4.*

$$\begin{array}{r} 11am - 7yd + mn \\ - 5am - 2yd - 7a \\ \hline 6am - 9yd + mn - 7a \end{array}$$

*Exam. 3.* —  $15a$  added to  $7a$  is  $-8a$ , by Art. 3. and  $14m$  added to  $-14m$  is *nothing* or 0, by Art. 5. therefore take no Notice of those Quantities in the Sum, and  $-16$  and  $y$  being different Quantities set them down by Art. 6. so is  $-8a - 16 + y$  the Sum required.

*Exam. 4.*  $11am$  added to  $-5am$  is  $6am$ , by Art. 3. and  $-7yd$  added to  $-2yd$  is  $-9yd$ , by Art. 1. But  $mn$  and  $-7a$  being different Quantities set them down by Art. 6. and  $6am - 9yd + mn - 7a$  is the Sum required.

*Exam. 5.*

$$\begin{array}{r} \text{To} \quad 4a - 17y + 15ap \\ \text{Add} \quad -2ap + 3a - 2y \\ \hline \text{Sum} \quad 13ap + 7a - 19y \end{array}$$

*Exam. 6.*

$$\begin{array}{r} -7m + 15 + 4a \\ -4a - 11 + 8m \\ \hline m + 4 \end{array}$$

*Exam. 5.* —  $2ap$  added to  $15ap$  is  $13ap$ , by Art. 3. and  $3a$  added to  $4a$  is  $7a$ , by Art. 1. and  $-2y$  added to  $-17y$  is  $-19y$ , by Art. 1. hence  $13ap + 7a - 19y$  is the Sum required.

*Exam. 6.* —  $7m$  added to  $8m$ , the Sum is  $m$ , by Art. 3.  $15$  added to  $-11$ , the Sum is  $4$ , by Art. 3. and  $4a$  added to  $-4a$ , the Sum is 0, or *nothing*, by Art. 5. hence  $m + 4$  is the Sum required.

In these two Examples the same Quantities are not set under one another, to shew the Learner that however they are placed, if the Quantities are alike, they are to be added as if they stood one under the other.

The more perfectly Addition is understood, the easier it will render the Work of Subtraction.

## SUBTRACTION.

# SUBTRACTION,

7. **I**S performed by one general Rule; change all the Signs of those Quantities which are to be subtracted, or suppose them in the Mind to be changed, then add these Quantities to the others, according to the several Rules of Addition, which will be the Difference or Remainder required.

I would advise the Learner to take out the Examples, and put down those Quantities which are to be subtracted with contrary Signs, to those they have in the Examples; that is, making those *affirmative* which are *negative*, and those *negative* which are *affirmative*, and then proceed as directed in the general Rule.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
From	5 <i>a</i>	7 <i>m</i>	— 5 <i>y</i>	— 8 <i>z</i>
Subtract	<u>3 <i>a</i></u>	<u>2 <i>m</i></u>	<u>— 2 <i>y</i></u>	<u>— 4 <i>z</i></u>
Remains	2 <i>a</i>	5 <i>m</i>	— 3 <i>y</i>	— 4 <i>z</i>

*Exam. 1.* Here 3 *a* the Quantity to be subtracted has the Sign +, which being made or supposed to be made —, then by the general Rule, 5 *a* is to be added to — 3 *a*, the Sum of which is 2 *a*, by Art. 3. and this is the Remainder required.

*Exam. 2.* In the same Manner 2 *m* being supposed to have the Sign — prefixed to it, then by the general Rule, 7 *m* is to be added to — 2 *m*, the Sum of which is 5 *m*, by Art. 3. and this is the Remainder required.

*Exam. 3.* And if we suppose — 2 *y* to be 2 *y*, or + 2 *y*, then by the general Rule, — 5 *y* added to + 2 *y*, the Sum is — 3 *y*, by Art. 3. and this is the Remainder required.

*Exam. 4.* If we suppose — 4 *z* to be 4 *z*, or + 4 *z*, then by the general Rule, if — 8 *z* is added to 4 *z*, the Sum is — 4 *z*, by Art. 3. and this is the Remainder required.

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
From	14 <i>m n</i>	— 7 <i>y d</i>	— 5 <i>y x</i>	4 <i>a y</i>
Subtract	<u>— 2 <i>m n</i></u>	<u>+ 5 <i>y d</i></u>	<u>+ 3 <i>y x</i></u>	<u>— 3 <i>a y</i></u>
Remains	16 <i>m n</i>	— 12 <i>y d</i>	— 8 <i>y x</i>	7 <i>a y</i>

*Exam. 5.* The Sign of 2 *m n* being —, if we suppose it +, then by the general Rule, 14 *m n* added to 2 *m n*, the Sum is 16 *m n*, by Art. 1. the Remainder required.

*Exam. 6.* If we suppose  $5yd$  to be  $-5yd$ , then by the general Rule,  $-7yd$  added to  $-5yd$ , the Sum is  $-12yd$ , by Art. 1. the Remainder required.

*Exam. 7.* By supposing  $3yx$  to be  $-3yx$ , then by the general Rule,  $-5yx$  added to  $-3yx$ , the Sum is  $-8yx$ , by Art. 1. the Remainder required.

*Exam. 8.* And if we suppose  $-3ay$  to be  $3ay$ , then by the general Rule,  $4ay$  added to  $3ay$ , the Sum is  $7ay$ , by Art. 1. the Remainder required.

From	$5am$	$-ay$	$-7ad$	$5yd$
Subtract	$-am$	$-5ay$	$+ad$	$yd$
Remains	$6am$	$4ay$	$-8ad$	$4yd$

The Truth of Subtraction may be proved as in common Arithmetic, by adding the Remainder to the Quantity which is subtracted, and if their Sum is the same as that from which the Quantity was subtracted, the Work is true, otherwise it is erroneous.

Thus in the four last Examples  $6am$  added to  $-am$ , the Sum is  $5am$ .

And  $4ay$  added to  $-5ay$ , the Sum is  $-ay$ .

And  $-8ad$  added to  $ad$ , the Sum is  $-7ad$ .

And  $4yd$  added to  $yd$ , the Sum is  $5yd$ . And in the same Manner may the other Examples be proved.

8. If two or more Quantities connected by the Signs  $+$  or  $-$ , are to be subtracted from other like Quantities connected by the Signs  $+$  or  $-$ , it is done in the same Manner, only taking due Care to connect the remaining Quantities with their proper Signs, as was done in the Addition of compound Quantities.

	<i>Exam. 9.</i>	<i>Exam. 10.</i>	<i>Exam. 11.</i>
From	$12a + 7b$	$7ma + 5y$	$-5zy - 2am$
Take	$3a + 2b$	$6ma + 4y$	$3zy - 4am$
Remains	$9a + 5b$	$ma + y$	$-8zy - 6am$

*Exam. 9.* By supposing  $3a$  to be  $-3a$ , then  $-3a$  added to  $12a$  the Sum is  $9a$ , by Art. 3. and again, supposing  $2b$  to be  $-2b$ , then  $-2b$  added to  $7b$  the Sum is  $5b$  by the same, and connecting these Quantities we have  $9a + 5b$ , the Remainder required.

*Exam. 10.*  $6ma$  being supposed negative, or to be  $-6ma$ , then  $-6ma$  added to  $7ma$  the Sum is  $ma$ , and  $4y$  being



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being supposed to be  $-4y$ , then  $-4y$  added to  $5y$  the Sum is  $y$ , hence  $ma + y$  is the Remainder required.

*Exam. 11.*  $3zy$  being supposed to be  $-3zy$ , and adding this to  $-5zy$  the Sum is  $-8zy$ , by Art. 1. and  $4am$  being supposed to be  $-4am$ , by adding that to  $-2am$  the Sum by Art. 1. is  $-6am$ , hence  $-8zy - 6am$  is the Remainder required.

	<i>Exam. 12.</i>	<i>Exam. 13.</i>	<i>Exam. 14.</i>
From	$14a - 5y$	$-7mn + 2yd$	$2a + m$
Take	$-3a - 5y$	$-3mn + 3yd$	$5a - 7$
Remains	$17a$	$-4mn - yd$	$-3a + m + 7$

*Exam. 12.* The  $-3a$  being supposed by the general Rule to be  $3a$ , and adding that to  $14a$  the Sum is  $17a$ , by Art. 1. and the  $-5y$  being supposed to be  $5y$ , if we add  $5y$  to  $-5y$ , the Sum is a *Cypher*, or *nothing*, by Art. 5. hence  $17a$  is the Remainder required.

*Exam. 13.* The  $-3mn$  being supposed to be  $3mn$ , then by adding  $3mn$  to  $-7mn$  the Sum is  $-4mn$ , by Art. 3. and  $3yd$  being supposed to be  $-3yd$ , and adding  $-3yd$  to  $2yd$  the Sum is  $-yd$ , by Art. 3. hence  $-4mn - yd$  is the Remainder required.

*Exam. 14.* The  $5a$  being supposed to be  $-5a$ , if that is added to  $2a$  the Sum is  $-3a$ , by Art. 3. but the  $m$  and  $7$  being different Quantities, set them down by Art. 6. only take particular Care to change the Sign of  $7$ , according to the general Rule for Subtraction, then will  $-3a + m + 7$  be the Remainder required.

	<i>Exam. 15.</i>	<i>Exam. 16.</i>	<i>Exam. 17.</i>
From	$-am + y$	$15yd + 20$	$14d + 7 - a$
Subtract	$+am + y$	$-3yd - 16$	$-d + 7 - 8a$
Remains	$-2am$	$18yd + 36$	$15d + 7a$

The Truth of these Operations is proved in the same Manner as in Subtraction of simple Quantities, by adding the Remainder to the Quantity which is subtracted, and observing if that Sum is the same, and has the same Signs, with those Quantities from which the Subtraction was made. Thus,

*Exam. 15.*  $-2am$  added to  $am$ , the Sum is  $-am$ , by Art. 3. to which connecting  $y$  with the Sign  $+$ , we find that by adding  $-2am$  to  $am + y$ , the Sum is  $-am + y$ , the Quantity from which the Subtraction was made.

*Exam. 16.* If  $18 y d$  is added to  $- 3 y d$  the Sum is  $15 y d$ , by Art. 3. and  $36$  added to  $- 16$  the Sum is  $20$  by the same, hence the Sum of  $18 y d + 36$  added to  $- 3 y d - 16$  is  $15 y d + 20$ , the Quantity from which the Subtraction was made.

*Exam. 17.* By adding  $15 d$  to  $- d$  the Sum is  $14 d$ , by Art. 3. and by adding  $7 a$  to  $- 8 a$  the Sum is  $- a$ , by the same, to which putting down the  $7$ , there being no Quantity to be added to that, hence  $15 d + 7 a$  added to  $- d + 7 - 8 a$  the Sum is  $14 d + 7 - a$ , the Quantity from which the Subtraction was made.

But if the Quantities to be subtracted are unlike those from which the Subtraction is to be made, set down these with the same Signs and Co-efficients they have in the Example, after which place the Quantities to be subtracted with their Co-efficients, but change their Signs.

	<i>Exam. 18.</i>	<i>Exam. 19.</i>	<i>Exam. 20.</i>
From	$2 a$	$- 3 y$	$5 m$
Take	$d$	$2 a$	$- 2 y$
Remains	$2 a - d$	$- 3 y - 2 a$	$5 m + 2 y$

*Exam. 18.* Having put down  $2 a$ , after which put  $- d$ , the Quantity to be subtracted being  $+ d$ , and  $2 a - d$  is the Remainder required.

*Exam. 19.* Having put down  $- 3 y$ , to that connect  $2 a$  with the Sign  $-$ , so is  $- 3 y - 2 a$  the Remainder required. The 20th Example is done in the same Manner.

And if compound Quantities are to be subtracted from compound Quantities, but unlike, set down all the Quantities one after the other, but change the Signs of those Quantities which are to be subtracted, as in these Examples.

From	$2 a + 5 m$	$- 4 d + 2 p$
Take	$3 y - 2 d$	$- 5 a + 3 y$
Remains	$2 a + 5 m - 3 y + 2 d$	$- 4 d + 2 p + 5 a - 3 y$

Having wrote down  $2 a + 5 m$ , to that connect  $3 y$  with the Sign  $-$ , it being  $+$  in the Example, to which connect  $2 d$  with the Sign  $+$ , it being  $-$  in the Example.

In the other Example, having wrote down  $- 4 d + 2 p$ , to this connect  $5 a$  with the Sign  $+$ , it being  $-$  in the Example, to which connect  $3 y$  with the Sign  $-$ , it being  $+$  in the Example.

# MULTIPLICATION,

*In which there are three Cases.*

(9.) *Case 1.* **W**HEN the Signs of the Quantities to be multiplied, are both *affirmative*, or both *negative*, set or join the Letters together, and to them prefix the Sign  $+$ , which will be the Product required.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Multiply	$a$	$y$	$-z$	$-da$
By	$d$	$m$	$-a$	$-x$
Product	$da$	$my$	$az$	$dax$

*Exam. 1.* Having joined the Letters  $da$ , and each of them having the *affirmative* Sign, therefore, by the Rule,  $da$ , or  $+da$ , is the Product required.

*Exam. 2.* Having joined the Letters  $m$  and  $y$ , and each of them having the *affirmative* Sign, therefore, by the Rule,  $my$ , or  $+my$ , is the Product required.

*Exam. 3.* Having joined the Quantities  $z$  and  $a$ , and each of them having the same Sign, therefore, by the Rule,  $az$ , or  $-az$ , is the Product required.

*Exam. 4.* Having joined the Quantities  $da$  and  $x$ , and both having the same Sign, therefore  $dax$ , or  $+dax$ , is the Product required.

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
Multiply	$a$	$-am$	$-y$	$am$
By	$a$	$-d$	$-dp$	$an$
Product	$aa$	$amd$	$dpy$	$aman$

*Exam. 5.* Having joined  $aa$ , and both the Quantities being *affirmative*, therefore  $aa$  is the Product required.

*Exam. 6.* Having joined the Quantities  $am$  and  $d$ , and both having the Sign  $-$ , hence  $amd$  is the Product required.

*Exam. 7.* Having joined the Quantities  $y$  and  $dp$ , and because both have the same Sign, therefore  $dpy$  is the Product required.

*Exam. 8.* Having joined the Quantities  $am$  and  $an$ , and both having the same Sign, therefore  $aman$  is the Product required.

10. If the Multiplicand consists of two or more Quantities connected by the Signs  $+$  or  $-$ , then the Multiplier must be multiplied into each of those Quantities, prefixing to each particular Multiplication its proper Sign, which will give the Product. Thus,

	<i>Exam. 9.</i>	<i>Exam. 10.</i>	<i>Exam. 11.</i>
Multiply	$a + d$	$z + y$	$-mx - n$
By	$m$	$d$	$-p$
Product	$ma + md$	$dz + dy$	$pmx + pn$

*Exam. 9.* If we multiply  $a$  by  $m$ , the Product is  $ma$ , by Art. 9. and multiplying  $d$  by  $m$ , the Product is  $md$ , by Art. 9. but as  $m$  and  $d$  have both the affirmative Sign, therefore prefix the Sign  $+$  before  $md$ , and  $ma + md$  is the Product required.

*Exam. 10.* Multiplying  $z$  by  $d$ , the Product is  $dz$ , and multiplying  $y$  by  $d$ , the Product is  $dy$ , by Art. 9. but as  $d$  and  $y$  have both the Sign  $+$ , prefixing that Sign before  $dy$  we have  $dz + dy$ , the Product required.

*Exam. 11.* Multiplying  $-mx$  by  $-p$ , the Product is  $pmx$ , by Art. 9. and for the same Reason  $-n$  multiplied by  $-p$ , the Product is  $pn$ , then connecting  $pmx$  and  $pn$  with the Sign  $+$ , we have  $pmx + pn$ , the Product required.

	<i>Exam. 12.</i>	<i>Exam. 13.</i>	<i>Exam. 14.</i>
Multiply	$-m - y$	$-a - zy$	$ad + z$
By	$-d$	$-x$	$y$
Product	$dm + dy$	$ax + xzy$	$ady + yz$

*Exam. 12.* Multiplying  $-m$  by  $-d$ , the Product is  $md$ , by what was said at Example 11, and multiplying  $-d$  by  $-y$ , the Product is for the same Reason  $dy$ , and connecting  $dm$  and  $dy$  with the Sign  $+$ , we have  $dm + dy$ , the Product required.

*Exam. 13.* Multiplying  $-a$  by  $-x$ , we have  $ax$  for the Product, as in the last Example, and from multiplying  $-zy$  by  $-x$ , we have for the same Reason  $xzy$  for this Product, then connecting  $ax$  and  $xzy$  with the Sign  $+$ , we have  $ax + xzy$ , the Product required.

*Exam. 14.* Multiplying  $ad$  by  $y$ , the Product is  $ady$ , and multiplying  $z$  by  $y$ , the Product is  $yz$ ; but as the Signs of  $y$  and  $z$  are both alike, therefore prefixing the Sign  $+$  to  $yz$ , we have  $ady + yz$ , the Product required.

# MULTIPLICATION. 21

11. It may be proper to caution the Learner, that in Multiplication it is quite indifferent which Letter he places first, or last, for to multiply  $am$  by  $d$ , the Product is  $amd$ , or  $mda$ , or  $dma$ , or  $adm$ , or any of the different Positions in which three Letters can be placed; this will be more compleatly and fully understood when we come to apply the Science to the Solution of Problems: But, that the Learner may form some Idea of the Truth of this, suppose we were to multiply 3, 5, and 7 together, the Product will be the same in whatever Order these three Numbers are multiplied. Thus,

$$\begin{array}{r} 3 \\ 5 \\ \hline 15 \\ 7 \\ \hline 105 \end{array}$$

$$\begin{array}{r} 7 \\ 5 \\ \hline 35 \\ 3 \\ \hline 105 \end{array}$$

$$\begin{array}{r} 3 \\ 7 \\ \hline 21 \\ 5 \\ \hline 105 \end{array}$$

This Observation I advise the Learner to fix in his Mind, to prevent concluding he has done any of the following Examples erroneously, by happening to place the Letters different from what they are in the Book.

12. But if the Multiplier and Multiplicand consists of two or more Quantities, then begin and multiply the Multiplicand by any one Quantity in the Multiplier, according to the Directions in Art. 9. and 10. after that multiply the Multiplicand by another Quantity in the Multiplier, and put this Product under the other, and continue doing this till the Multiplicand has been multiplied by every Quantity in the Multiplier; then under these Products draw a Line, and add them together by the several Cases of Addition, and this will be the Product required.

*Exam. 1.*

Multiply  $a + b$

By

$m + n$

$ma + mb$  the Product of  $a + b$  multiplied by  $m$ , by Art. 10.

$na + nb$  the Product of  $a + b$ , multiplied by  $n$ , by the same.

$ma + mb + na + nb$  the Sum by Art. 6. the Product required.

Multiply

$$\begin{array}{r}
 \text{Multiply } m + y \\
 \text{By } a + d \\
 \hline
 am + ay \text{ the Product of } m + y \text{ multiplied by } a, \text{ by} \\
 \text{Art. 10.} \\
 md + yd \text{ the Product of } m + y \text{ multiplied by } d, \text{ by} \\
 \text{the same.} \\
 \hline
 am + ay + md + yd \text{ the Sum by Art. 6. the Pro-} \\
 \text{duct required.}
 \end{array}$$

$$\begin{array}{r}
 \text{Multiply } -a - d \\
 \text{By } -m - z \\
 \hline
 ma + md \text{ the Product of } -a - d \text{ multiplied by} \\
 -m, \text{ by Art. 10.} \\
 az + dz \text{ the Product of } -a - d \text{ multiplied by } -z, \\
 \text{by the same.} \\
 \hline
 ma + md + az + dz \text{ the Sum by Art. 6. the Product} \\
 \text{required.}
 \end{array}$$

$$\begin{array}{r}
 \text{Multiply } a + b \\
 \text{By } a + b \\
 \hline
 aa + ab \text{ the Product of } a + b \text{ multiplied by } a, \text{ by} \\
 \text{Art. 10.} \\
 ab + bb \text{ the Product of } a + b \text{ multiplied by } b, \text{ by the} \\
 \text{same.} \\
 \hline
 aa + 2ab + bb \text{ the Product of } a + b \text{ multiplied by} \\
 a + b.
 \end{array}$$

Now in the Addition of the last Example I observe there is  $ab$  in each of the two Lines, and there being no Co-efficient prefixt, *Unity* or 1 being then always understood to be the Co-efficient, hence 1  $ab$  added to 1  $ab$  is 2  $ab$ ; the other Quantities  $aa$  and  $bb$  are set down as in the former Examples, therefore  $aa + 2ab + bb$  is the Product required.

And in such Additions as these I recommend it to the Learner, before he begins to add, to examine the several Quantities, and see if the Letters in any two of them are alike, and if they are to collect them into one Sum, according to Art. 1 and 3; remembering, that though the Letters which compose the two Quantities are not in the same Order in each; yet if they are but the same Letters, and no more in one than there is in the other Quantity, they are the same, and may be added by Art. 1 and 3.

The

# MULTIPLICATION. 23

The four following Examples are for the Exercise of the Learner.

Multiply	$a + b$		$a + y$
By	$m + y$		$a + y$
	<hr/> $am + mb$		<hr/> $aa + ay$
	$ay + by$		$ay + yy$
Product	<hr/> $am + mb + ay + by$		<hr/> $aa + 2ay + yy$

Multiply	$y + m$		$-a - d$
By	$y + m$		$-a - d$
	<hr/> $yy + ym$		<hr/> $aa + ad$
	$ym + mm$		$aa + dd$
Product	<hr/> $yy + 2ym + mm$		<hr/> $aa + 2ad + dd$

13. *Case 2.* If there are Co-efficients or prefix Numbers, then multiply the Numbers as in common Arithmetic, and to their Products join the Products of the Quantities found by the last *Case*, Art. 9.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Multiply	$2a$	$5m$	$-7ad$	$-2y$
By	$3m$	$3y$	$-3m$	$-a$
Product	<hr/> $6am$	<hr/> $15my$	<hr/> $21adm$	<hr/> $2ay$

*Exam. 1.* The Product of the Co-efficients 2 and 3 is 6: the Product of  $a$  multiplied by  $m$  is  $am$ , the Signs being alike, joining these together it is  $6am$ , the Product required.

*Exam. 2.* The Product of 5 by 3 is 15: the Product of  $m$  by  $y$  is  $my$ , for the Signs are alike, and joining these together it is  $15my$ , the Product required.

*Exam. 3.* The Product of 7 by 3 is 21: the Product of  $a$  by  $d$  is  $adm$ , the Signs being alike, and joining the 21 and  $adm$  it is  $21adm$ , the Product required.

*Exam. 4.* The Product of 2, and 1 the Co-efficient of  $a$ , is 2, to which joining  $ay$ , the Product of  $a$  and  $y$ , it is  $2ay$ , the Product required, for the Signs of  $2y$  and  $a$  are alike, being both negative.

Multiply	$7am$	$6dz$	$-3yp$	$-2dz$
By	$2d$	$2a$	$-3y$	$-d$
Product	<hr/> $14amd$	<hr/> $12dza$	<hr/> $9yyp$	<hr/> $2ddz$

14. And

14. And if there are two or more Quantities with Co-efficients connected by the Signs  $+$  or  $-$ , to be multiplied by any Quantity and its Co-efficient, they are multiplied as in the last Article, only connecting the several particular Products together with their proper Signs, as was done at Art. 10.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
Multiply	$2a + 3b$	$3y + 5d$	$-2y - 2a$
By	$3m$	$5m$	$-3z$
Product	$6am + 9bm$	$15ym + 25dm$	$6yz + 6za$

*Exam. 1.* Multiplying  $2a$  by  $3m$  the Product is  $6am$ , by Art. 13. and then multiplying  $3m$  by  $3b$  the Product is  $9bm$ , by Art. 13. to which prefixing the *affirmative* Sign, as the Signs of  $3b$  and  $3m$  are alike, and  $6am + 9bm$  is the Product required.

*Exam. 2.* Multiplying  $3y$  by  $5m$  the Product is  $15ym$ , by Article 13. then multiplying  $5m$  by  $5d$  the Product is  $25dm$ , to which prefixing the affirmative Sign, as the Signs of  $5d$  and  $5m$  are alike, and  $15ym + 25dm$  is the Product required.

*Exam. 3.* Multiplying  $-2y$  by  $-3z$  the Product is  $6yz$ , by Art. 9 and 13. Again, multiplying  $-2a$  by  $-3z$  the Product is  $6az$ , for the Signs of  $2a$  and  $3z$  are alike, and connecting  $6yz$  and  $6az$  with the Sign  $+$ , we have  $6yz + 6za$ , the Product required.

	<i>Exam. 4.</i>	<i>Exam. 5.</i>	<i>Exam. 6.</i>
Multiply	$3m + 2y$	$-2d - 3m$	$-3y - 7my$
By	$6a$	$-4a$	$-2a$
Product	$18ma + 12ya$	$8da + 12ma$	$6ya + 14amy$

*Exam. 4.* Multiplying  $3m$  by  $6a$  the Product is  $18ma$ , and multiplying  $2y$  by  $6a$  the Product is  $12ya$ , and placing the Sign  $+$  before  $12ya$ , because the Signs of  $6a$  and  $2y$  are alike, we have  $18ma + 12ya$ , the Product required.

*Exam. 5.* Multiplying  $-2d$  by  $-4a$  the Product is  $8da$ , for the Signs of  $2d$  and  $4a$  are alike, being both negative, therefore  $8da$  or  $+8da$  is the Product of these Quantities. Now multiplying  $-4a$  by  $-3m$  the Product is  $12ma$ , to which prefixing the affirmative Sign, as  $3m$  and  $4a$  have the same Sign, both being negative, and we have  $8da + 12ma$ , the Product required.

*Exam.*



# MULTIPLICATION. 25

*Exam. 6.* The Product of  $-3y$  by  $-2a$  is  $6ya$ , or  $+6ya$ , and the Product of  $-7my$  by  $-2a$  is  $14amy$ , or  $+14amy$ , hence, for the Reason in the last Example,  $6ya + 14amy$  is the Product required.

Multiply $-3m - 2d$	Multiply $-2z - 3y$	Multiply $-4d - 5m$
By $-4a$	By $-4a$	By $-2b$
Product $12ma + 8da$	Product $8za + 12ay$	Product $8bd + 10mb$

15. And if there are two or more Quantities with Coefficients connected by the Signs  $+$  or  $-$ , to be multiplied by two or more Quantities with Coefficients connected in the same Manner, the Quantities are to be multiplied as at Art. 12. taking due Care to multiply the Coefficients, as has been taught Art. 14. Thus,

Multiply  $2a + 3b$   
 By  $3a + 5m$

---

$6aa + 9ab$  the Product of  $2a + 3b$  multiplied by  $3a$ , by Art. 14.  
 $10ma + 15bm$  the Product of  $2a + 3b$  multiplied by  $5m$ , by the same.

---

$6aa + 9ab + 10ma + 15bm$  the Product required, being the Sum of the two particular Products, which are added together by Art. 6.

Multiply  $3m + 5y$   
 By  $2a + 3n$

---

$6am + 10ay$  the Product of  $3m + 5y$  multiplied by  $2a$ , by Art. 14.  
 $9mn + 15yn$  the Product of  $3m + 5y$  multiplied by  $3n$ , by Art. 14.

---

$6am + 10ay + 9mn + 15yn$  the Product required, being the Sum of the two Products, which are added together by Art. 6.

Multiply  $2a + 3b$   
 By  $2a + 2b$

---

$4aa + 6ab$  the Product of  $2a + 3b$  multiplied by  $2a$ , by Art. 14.  
 $4ab + 6bb$  the Product of  $2a + 3b$  multiplied by  $2b$ , by the same.

---

$4aa + 10ab + 6bb$  the Product required. In this Addition

E

Addition the Reader is to observe that in one Line there is  $6ab$ , and in the other Line there is  $4ab$ , which two Quantities added together the Sum is  $10ab$ , by Art. 1. but the  $4aa$  and  $6bb$  being different Quantities, they are set down by Art. 6. hence the Product of  $2a + 3b$  multiplied by  $2a + 2b$ , is  $4aa + 10ab + 6bb$ .

Multiply	$3a + 7b$		$3a + 2b$
By	$2a + 5b$		$a + 4b$
	$6aa + 14ba$		$3aa + 2ba$
	$15ab + 35bb$		$12ba + 8bb$
Product	$6aa + 14ba + 15ab + 35bb$		$3aa + 14ba + 8bb$

16. *Case 3.* When the Signs of the two Quantities to be multiplied are one affirmative and the other negative, then multiply the Quantities as before directed, but to their Product prefix —, or the negative Sign.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Multiply	$-b$	$-a$	$-am$	$-dm$
By	$a$	$d$	$y$	$z$
Product	$-ba$	$-ad$	$-amy$	$-dmz$

*Exam. 1.* The Product of  $b$  by  $a$  is  $ba$ , for Multiplication is only joining the Letters, but as the Sign of  $b$  is —, and that of  $a$  is +, therefore to  $ba$  prefix the Sign —, so is  $-ba$  the Product required.

*Exam. 2.* The Product of  $a$  by  $d$  is  $ad$ , but as the Signs of  $a$  and  $d$  are different, therefore prefix the Sign — to  $ad$ , and  $-ad$  is the Product required.

*Exam. 3.* The Product of  $am$  by  $y$  is  $amy$ , but as the Signs of  $am$  and  $y$  are different, therefore prefix the Sign — to  $amy$ , so is  $-amy$  the Product required.

*Exam. 4.* The Product of  $dm$  by  $z$  is  $dmz$ , but as the Signs of  $dm$  and  $z$  are different, therefore prefix the Sign — to  $dmz$ , so is  $-dmz$  the Product required.

This Operation being the same as at Art. 9. taking Care to make the Sign of the Product —, I shall only subjoin the following Examples for the Exercise of the Learner.

Multiply

Multiply	$xm$	$-ym$	$-ax$	$ma$
By	$-y$	$d$	$dy$	$-p$
Product	$-xmy$	$-ymd$	$-axdy$	$-map$

17. And if two or more Quantities with Co-efficients are to be multiplied into any one Quantity with a different Sign, they are multiplied as at Art. 14. taking Care of the Signs arising in the Product, according to Art. 9. and 16.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
Multiply	$-3a - 2z$	$2y + 5d$	$-5m - 2y$
By	$3m$	$-3a$	$3d$
Product	$-9am - 6zm$	$-6ay - 15ad$	$-15md - 6dy$

*Exam. 1.* Multiplying  $3a$  by  $3m$ , the Product by Art. 13. is  $9am$ , but as  $3m$  has the Sign  $+$  prefixed to it, and  $3a$  has the Sign  $-$  prefixed to it, therefore to the Product  $9am$  prefix the Sign  $-$ , by Art. 16. Again,  $2z$  multiplied by  $3m$  the Product is  $6zm$ , but as  $2z$  has the Sign  $-$  to it, and  $3m$  the Sign  $+$ , therefore to  $6zm$  prefix the Sign  $-$ , by Art. 16. and  $-9am - 6zm$  is the Product required.

*Exam. 2.* Multiplying  $2y$  by  $3a$ , the Product is  $6ay$ , but as the Sign of  $2y$  is  $+$ , and that of  $3a$  is  $-$ , therefore to  $6ay$  prefix the Sign  $-$ , by Art. 16. Then multiplying  $5d$  by  $3a$  the Product is  $15ad$ , but as the Sign of  $5d$  is  $+$ , and that of  $3a$  is  $-$ , therefore prefix the Sign  $-$  to  $15ad$  by Art. 16. and  $-6ay - 15ad$  is the Product required.

*Exam. 3.* Multiplying  $5m$  by  $3d$ , the Product is  $15md$ , but as the Sign of  $5m$  is  $-$ , and that of  $3d$  is  $+$ , therefore prefix the Sign  $-$  to  $15md$ . Again, multiplying  $2y$  by  $3d$ , the Product is  $6yd$ , but because the Signs of  $2y$  and  $3d$  are different, therefore prefix the Sign  $-$  to  $6dy$ , and  $-15md - 6dy$  is the Product required.

Examples for the Exercise of the Learner.

Multiply	$-2a - 3b$	$3m + 7y$	$-5y - 3b$
By	$4z$	$-2d$	$3m$
Product	$-8az - 12bz$	$-6md - 14dy$	$-15ym - 9bm$

18. And if two or more Quantities with Co-efficients are to be multiplied by two or more Quantities with Co-efficients, if their Signs are unlike, yet they are multiplied as at Art. 15. taking due Care of the Signs of the Product, by Art. 9. and 16.

Multiply  $— 3a — 2m$

By  $4b + 6y$

$— 12ab — 8bm$  the Product of  $— 3a — 2m$  multiplied by  $4b$ , by Art. 17.

$— 18ay — 12ym$  the Product of  $— 3a — 2m$  multiplied by  $6y$ , by Art. 17.

$— 12ab — 8bm — 18ay — 12ym$  the Product required, being the Sum of the two particular Products, which are added by Art. 6.

Multiply  $5y + 3m$

By  $— 7d — 3a$

$— 35yd — 21dm$  the Product of  $5y + 3m$  multiplied by  $— 7d$ , by Art. 17.

$— 15ay — 9am$  the Product of  $5y + 3m$  multiplied by  $— 3a$ , by Art. 17.

$— 35yd — 21dm — 15ay — 9am$  the Product required, being the Sum of the two particular Products, which are added by Art. 6.

Multiply  $2a + 3b$

By  $— 2a — 3b$

$— 4aa — 6ab$  the Product of  $2a + 3b$  multiplied by  $— 2a$ , by Art. 17.

$— 6ab — 9bb$  the Product of  $2a + 3b$  multiplied by  $— 3b$ , by Art. 17.

$— 4aa — 12ab — 9bb$  the Product required: For in this Addition the Reader may observe that there is  $— 6ab$  in each of the two particular Products, which being added together by Art. 1. make  $— 12ab$ , but  $— 4aa$  and  $— 9bb$  being different Quantities, they must be placed separate from one another. There are Examples of this Kind at Art. 15.

19. It may be for the Learner's Advantage to be put in Mind, that if any Algebraic Quantities are to be multiplied by a pure Number, that then this Number is to be multiplied into every one of the Co-efficients of the other Quantities, in all Respects as before, and to each particular Product set or join that Quantity whose Co-efficient was multiplied. Thus,

*Exam.*

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
Multiply	$2a + 3b$	$-3m - 4d$	$4d + 3y$
By	$6$	$7$	$9$
Product	$12a + 18b$	$-21m - 28d$	$36d + 27y$

*Exam. 1.* Multiplying 6 by 2 the Product is 12, to which joining  $a$  it is  $12a$ , then multiplying 3 by 6 it is 18, to which joining  $b$  it is  $18b$ , and because 6 and  $3b$  have both the Sign +, therefore by Art. 9. prefix the Sign + to  $18b$ , so is  $12a + 18b$  the Product required.

*Exam. 2.* Multiplying 3 by 7 it is 21, to which joining  $m$  it is  $21m$ , but as the Sign of  $3m$  is —, and that of 7 is +, therefore by Art. 16. prefix the Sign — to  $21m$ , and it is  $-21m$ . Again, multiplying 4 by 7 it is 28, to which joining  $d$  it is  $28d$ , but as the Signs of  $4d$  and 7 are likewise unlike, therefore to  $28d$  prefix the Sign —, and  $-21m - 28d$  is the Product required.

*Exam. 3.* Multiplying 4 by 9 the Product is 36, to which joining  $d$  it is  $36d$ , and because the Signs of  $4d$  and 9 are alike, therefore it will be  $36d$ , or  $+36d$ ; and multiplying 9 by  $3y$  the Product will be  $27y$ , to which must be prefixt the Sign +, because  $3y$  and 9 have the same Sign, so is  $36d + 27y$  the Product required.

Examples wherein all the three Cases of Multiplication are promiscuously used.

$$\begin{array}{r}
 \text{Multiply} \quad 2a - 3b \\
 \text{By} \quad \quad \quad 5m + 2y \\
 \hline
 10ma - 15mb \\
 4ay - 6yb \\
 \hline
 \text{Product} \quad 10ma - 15mb + 4ay - 6yb
 \end{array}$$

The  $2a$  being multiplied by  $5m$  the Product is  $10ma$ , by Art. 13. and  $-3b$  being multiplied by the same  $5m$ , the Product is  $-15mb$ , by Art. 16. and 17.

And  $2a$  being multiplied by  $2y$  the Product is  $4ay$ , by Art. 13. and  $-3b$  multiplied by  $2y$  the Product is  $-6yb$ , by Art. 16. and 17.

Now draw the Line and begin to add them, and because the Quantities are all different, they are added by Art. 6. and therefore the Product will be  $10ma - 15mb + 4ay - 6yb$ .

$$\begin{array}{r}
 \text{Multiply} \quad -7m + 2a \\
 \text{By} \quad \quad \quad 3y - 4n \\
 \hline
 \quad \quad -21my - 6ay \\
 \quad \quad \quad 28mn - 8na \\
 \hline
 \text{Product} \quad -21my + 6ay + 28mn - 8na
 \end{array}$$

The  $-7m$  multiplied by  $3y$  the Product is  $-21my$ , by Art. 16 and 17. and  $2a$  multiplied by  $3y$  the Product is  $6ay$ , by Art. 13.

And  $-7m$  multiplied into  $-4n$  the Product is  $28mn$ , by Art. 13. and  $-4n$  multiplied by  $2a$  the Product is  $-8na$ , by Art. 16 and 17.

Now begin the Addition, and because the Quantities are all different, they are added by Art. 6. and the Product is  $-21my + 6ay + 28mn - 8na$ .

$$\begin{array}{r}
 \text{Multiply} \quad 2a + 3b \\
 \text{By} \quad \quad \quad 2a - 3b \\
 \hline
 \quad \quad 4aa + 6ab \\
 \quad \quad -6ab - 9bb \\
 \hline
 \text{Product} \quad 4aa - 9bb
 \end{array}$$

Multiplying  $2a$  by  $2a$  the Product is  $4aa$ , and multiplying  $3b$  by  $2a$  the Product is  $6ab$ .

And multiplying  $2a$  by  $-3b$  the Product is  $-6ab$ , because the Signs of the two Quantities are unlike, and for the same Reason the Product of  $3b$  by  $-3b$  is  $-9bb$ .

Now begin the Addition, and I observe in the first Line there is  $+6ab$  or  $6ab$ , but in the second Line there is  $-6ab$ , now because the Co-efficients are equal, the Quantities alike, but the Signs being contrary, therefore by Art. 5. these Quantities will destroy one another, then putting down the  $4aa$  and  $-9bb$ , by Art. 6. we have  $4aa - 9bb$ , the Product required.

$$\begin{array}{r}
 \text{Multiply} \quad 7m + 4a \\
 \text{By} \quad \quad \quad -3a + 5 \\
 \hline
 \quad \quad -21ma - 12aa \quad \text{the Product of } 7m + 4a \text{ multi-} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \text{plied by } -3a. \\
 \quad \quad \quad 35m - 20a \quad \text{the Product of } 7m - 4a \text{ multi-} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \text{plied by } 5, \text{ by Art. 19.} \\
 \hline
 \text{Product} \quad -21ma - 12aa + 35m + 20a
 \end{array}$$

Multiply

$$\begin{array}{r}
 \text{Multiply } 5a + b \\
 \text{By } 2a + 3b \\
 \hline
 10aa + 2ab \text{ the Product of } 5a + b \text{ multiplied by } 2a. \\
 15ab + 3bb \text{ the Product of } 5a + b \text{ multiplied by } 3b. \\
 \hline
 \text{Product } 10aa + 17ab + 3bb.
 \end{array}$$

# DIVISION,

*In which there are four Cases.*

20. *Case 1.* **W**HEN the Signs of the Quantities to be divided are both affirmative, or both negative, reject all those Quantities in the Dividend and Divisor that are alike, and set down the Remainder, prefixing to it the Sign +, which will be the Quotient required.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Divide	$ab$	$dm$	$mn$	$ap$
By	$a$	$d$	$m$	$p$
Quotient	$\frac{b}{a}$	$\frac{m}{d}$	$\frac{n}{m}$	$\frac{a}{p}$

*Exam. 1.* Because  $a$  is in the Dividend and Divisor, reject it, and  $b$  being only left, it is the Quotient sought, and is to have the Sign +, because the Signs of  $ab$  and  $a$  are alike.

*Exam. 2.* Because  $d$  is in the Dividend and Divisor, reject it, and there being only  $m$  left, it is the Quotient sought, which must have the Sign +, because the Signs of  $dm$  and  $d$  are alike.

*Exam. 3.* Because  $m$  is in the Dividend and Divisor, reject it, and  $n$  being only left, I write it down for the Quotient sought, which must have the Sign +, because  $mn$  and  $m$  have the same Sign.

*Exam 4.* Because  $p$  is in the Dividend and Divisor, I reject it, and place down  $a$ , the Quantity left, for the Quotient sought, which must have the Sign +, for the Signs of  $ap$  and  $p$  are alike.

*Exam.*

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
Divide	$a m d$	$- a p y$	$m d a$	$- m y z$
By	$a m$	$- p y$	$m a$	$- z$
Quotient	$d$	$a$	$d$	$y m$

*Exam. 5.* Because  $a m$  is in the Dividend and Divisor, reject it, and place down  $d$  for the Quotient, which must have the affirmative Sign, for the Signs of  $a m d$  and  $a m$  are alike.

*Exam. 6.* Because  $p y$  is in the Dividend and Divisor, I reject it, and place down  $a$ , the remaining Quantity, for the Quotient, which must have the affirmative Sign, for the Signs of  $a p y$  and  $p y$  are alike.

*Exam. 7.* Because  $m a$  is in the Dividend and Divisor, I reject it, and place down  $d$  for the Quotient, which must have the Sign  $+$ , because the Signs of  $m d a$  and  $m a$  are alike.

*Exam. 8.* Because  $z$  is in the Dividend and Divisor, I reject it, and place down  $y m$ , or  $m y$ , which is the same thing, for the Quotient sought, and must have the Sign  $+$ , because the Signs of  $m y z$  and  $z$  are alike.

Divide	$a p z$	$- m n d$	$- a b c$	$a b d y$
By	$a z$	$- m d$	$- c$	$a y$
Quotient	$p$	$n$	$a b$	$b d$

The Truth of these Operations in Division may be proved like those in Arithmetic, for the Quotient and Divisor being multiplied, the Product will be the Dividend if the Work is true; thus in the second Example of the last four, by multiplying  $n$  the Quotient into  $- m d$  the Divisor, the Product is  $m d n$ , or  $m n d$ , to which must be prefixt the Sign  $-$ , by Art. 16. because the Signs of  $m d$  and  $n$  are unlike, hence the Product with its Sign is  $- m n d$ , the given Dividend.

And in the last Example, if we multiply  $b d$  the Quotient by  $a y$  the Divisor, the Product is  $b d a y$ , or  $a b d y$ , which is the same thing, by Art. 11. and this Quantity must have the affirmative Sign, by Art. 9. for the Signs of  $b d$  and  $a y$  are alike, hence  $+ a b d y$ , or  $a b d y$ , is the Product with its Sign, the same as the given Dividend: And so of any other Example.

21. But if all the Quantities in the Divisor are not to be found in the Dividend, then you must only reject those Quantities in  
the



the Dividend and Divisor that are alike, placing down the remaining Quantities of the Dividend, and under them those of the Divisor that are not rejected by this Rule; which will be the Quotient sought, and stand like a Vulgar Fraction in common Arithmetic.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Divide	$a m b$	$— m n d z$	$— d a y p$	$p n q r$
By	$\frac{a y}{m b}$	$\frac{— m n a}{d z}$	$\frac{— d p z}{a y}$	$\frac{p a d}{n q r}$
Quotient	$\frac{m b}{y}$	$\frac{d z}{a}$	$\frac{a y}{z}$	$\frac{n q r}{a d}$

*Exam. 1.* Because  $a$  is in the Dividend and Divisor, reject it, and place down  $m b$  the remaining Part of the Dividend, under which drawing a Line, and place  $y$  the remaining Part of the Divisor, so will  $\frac{m b}{y}$  be the Quotient sought, which must have the Sign  $+$ , by Art. 20. as the Signs of the Quantities to be divided are alike.

*Exam. 2.* Because  $m n$  is in the Dividend and Divisor, reject it, and place down  $d z$  the remaining Part of the Dividend, under which drawing a Line, and place  $a$  the remaining Part of the Divisor, so is  $\frac{d z}{a}$  the Quotient required, and it must have the Sign  $+$ , by Art. 20. as the Signs of the Quantities to be divided are alike.

*Exam. 3.* Because  $d p$  is in both Dividend and Divisor, reject it, and write down  $a y$  the remaining Part of the Dividend, under which place  $z$  the remaining Part of the Divisor, as in the two former Examples, so is  $\frac{a y}{z}$ , or  $+\frac{a y}{z}$ , the Quotient required, for the Signs of the two Quantities to be divided are alike.

*Exam. 4.* Because  $p$  is in both Dividend and Divisor, reject it, and write down  $n q r$  the remaining Part of the Dividend, under which place  $a d$  the remaining Part of the Divisor, and  $\frac{n q r}{a d}$  is the Quotient required, which will be affirmative by Art. 20. because the Signs of  $p n q r$  and  $p a d$  are alike.

Divide	$\frac{-apqn}{-anm}$	$\frac{adz}{ap}$	$\frac{mnd}{ma}$	$\frac{-yzdb}{-yza}$
By	$\frac{pq}{m}$	$\frac{dz}{p}$	$\frac{nd}{a}$	$\frac{db}{a}$
Quotient	$\frac{pq}{m}$	$\frac{dz}{p}$	$\frac{nd}{a}$	$\frac{db}{a}$

These Operations are proved as at Art. 20. by multiplying the Quotient by the Divisor; for in the last Example the Quotient is  $\frac{db}{a}$ , which is a Fraction: the Divisor is  $-yza$ , which by the Rule of Vulgar Fractions in common Arithmetic is made this improper Fraction  $-\frac{yza}{1}$ , then the two Fractions to be multiplied

are  $\frac{db}{a}$ , and  $-\frac{yza}{1}$ , multiplying the Numerators we have  $db y z a$  for the new Numerator, and multiplying the Denominators we have  $a$  for the Denominator, hence the Product is this Fraction  $\frac{db y z a}{a}$ , but as  $\frac{yza}{1}$  has the *negative* Sign, and  $\frac{db}{a}$  has the affirmative Sign, therefore by Art. 16. prefix the Sign  $-$  to  $\frac{db y z a}{a}$  and it is  $-\frac{db y z a}{a}$ , the Product with its true Sign:

But in this Fraction as  $-db y z a$  is to be divided by  $a$ , rejecting  $a$  both in Dividend and Divisor by Art. 20. we have  $-db y z$  or  $-yzdb$ , the same with the Dividend in the given Example; in like Manner may the others be proved.

22. And if there are two or more Quantities connected by the Signs  $+$  or  $-$  to be divided by any single Quantity, every Quantity in the Dividend must be divided by the Divisor, setting down the particular Quotients, as at Art. 20. which must be connected by the Sign  $+$  when the Signs of the Quantities to be divided are both alike.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
Divide	$ab + am$	$md + mz$	$-da - dpq$
By	$a$	$m$	$-d$
Quotient	$\frac{b + m}{1}$	$\frac{d + z}{1}$	$\frac{a + pq}{1}$

*Exam. 1.* Dividing  $ab$  by  $a$  the Quotient is  $b$ , by Art. 20. and dividing  $am$  by  $a$  the Quotient is  $m$ , by the same Art. 20.

as  $a m$  and  $a$  have both the affirmative Sign, therefore to  $m$  prefix the Sign  $+$ , so is  $b + m$  the Quotient required.

*Exam. 2.*  $m d$  being divided by  $m$  the Quotient is  $d$ , by Art. 20. and dividing  $m z$  by  $m$  the Quotient is  $z$ , to which prefixing the Sign  $+$ , as  $m z$  and  $m$  have both the same Sign, we have  $d + z$  for the Quotient required.

*Exam. 3.*  $d a$  being divided by  $d$  the Quotient is  $a$ , and because  $d a$  and  $a$  have both the negative Sign, or the Signs are alike, therefore  $a$  must have the Sign  $+$ , whence it is  $+ a$  or  $a$ , and dividing  $- d p q$  by  $- d$  the Quotient is  $p q$ , to which must be prefixed the Sign  $+$ , for the Signs of  $d p q$  and  $d$  are alike, hence we have  $a + p q$  for the Quotient required.

	<i>Exam. 4.</i>	<i>Exam. 5.</i>	<i>Exam. 6.</i>
Divide	$- a b - a m$	$b m + b n$	$- z y p - z y a$
By	$- a$	$b$	$- z y$
Quotient	$b + m$	$m + n$	$p + a$

*Exam. 4.* Dividing  $- a b$  by  $- a$  the Quotient is  $b$ , by Art. 20. and it must be  $+ b$  or  $b$ , as the Signs of  $a b$  and  $a$  are alike: then dividing  $- a m$  by  $- a$  the Quotient is  $m$ , by Art. 20. because the Signs of  $a m$  and  $a$  are alike, then connecting  $b$  and  $m$  with the Sign  $+$ , and  $b + m$  is the Quotient required.

*Exam. 5.* Dividing  $b m$  by  $b$  the Quotient is  $m$ , by Art. 20. and dividing  $b n$  by  $b$  the Quotient is  $n$ , and as  $b n$  and  $b$  have the same Sign, therefore prefix the Sign  $+$  to  $n$ , so is  $m + n$  the Quotient required.

*Exam. 6.* Dividing  $- z y p$  by  $- z y$  the Quotient is  $p$ , by Art. 20. and dividing  $- z y a$  by  $- z y$  the Quotient is  $a$ , to which prefixing the Sign  $+$ , for the Signs of  $z y a$  and  $z y$  are alike, we have  $p + a$  the Quotient required.

Divide	$- d n z - z a d$	$a m + a d$	$- d y - d z$
By	$- d z$	$a$	$- d$
Quotient	$n + a$	$m + d$	$y + z$

The Truth of these Operations are proved by multiplying the Quotient by the Divisor; if that Product agrees with the Dividend in its Quantities and Signs the Work is true, otherwise not. Now in the last Example the Quotient  $y + z$ , and the Divisor  $- d$ , being multiplied together by Art. 14. they produce  $- d y - d z$ , the given Dividend.

23. *Case 2.* When the Signs of the Quantities to be divided are one *affirmative* and the other *negative*, find the Quotient of the Quantities as before ; but to them prefix the *negative* Sign, or Sign —.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Divide	$-am$	$-mnp$	$ayz$	$dmb$
By	$\frac{a}{-m}$	$\frac{m}{-np}$	$\frac{-ay}{-z}$	$\frac{-db}{-m}$
Quotient	$-m$	$-np$	$-z$	$-m$

*Exam. 1.* Because  $a$  is in both Dividend and Divisor, reject it, and place down  $m$  the remaining Part of the Dividend, but as the Signs of  $am$  and  $a$  are different, therefore to  $m$  prefix the Sign —, and it will be  $-m$ , the Quotient required.

*Exam. 2.* Because  $m$  is in the Dividend and Divisor, reject it, and place down  $np$  the remaining Part of the Dividend, but as the Signs of  $mnp$  and  $m$  are different, therefore to  $np$  prefix the Sign —, and it will be  $-np$ , the Quotient required.

*Exam. 3.* Because  $ay$  is in the Dividend and Divisor, reject it, and place down  $z$  the remaining Part of the Dividend, but as the Signs of  $ayz$  and  $ay$  are different, prefix the Sign — to  $z$ , and it will be  $-z$ , the Quotient required.

*Exam. 4.* Because  $db$  is in the Dividend and Divisor, reject it, and place down  $m$  the remaining Part of the Dividend ; but the Signs of the Quantities that are divided being different, to  $m$  prefix the Sign —, and it will be  $-m$ , the Quotient required.

Divide	$-mnp$	$-mnp$	$dyp$	$dab$
By	$\frac{m}{-np}$	$\frac{mn}{-p}$	$\frac{-dy}{-p}$	$\frac{-db}{-a}$
Quotient	$-np$	$-p$	$-p$	$-a$

24. And if there are two or more Quantities connected by the Signs + or —, to be divided by any single Quantity, the Operation is the same as at Art. 22. only taking due Care, when the Signs of the Quantities to be divided are different, to prefix the Sign — before those Quotients.

*Exam.*

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
Divide	$-mn - md$	$ad + ab$	$-dnz - dzy$
By	$m$	$-a$	$dz$
Quotient	$-n - d$	$-d - b$	$-n - y$

*Exam. 1.* Dividing  $-mn$  by  $m$  the Quotient is  $n$ , by Art. 20. but as the Signs of  $mn$  and  $m$  are different, therefore by Art. 23. I prefix the Sign  $-$  to  $n$ , and it is  $-n$ . And dividing  $-md$  by  $m$ , the Quotient is  $d$ ; but as the Signs of  $md$  and  $m$  are different, therefore by Art. 23. prefix the Sign  $-$  to  $d$ , hence  $-n - d$  is the Quotient required.

*Exam. 2.* Dividing  $ad$  by  $-a$  the Quotient is  $d$ , to which prefix the Sign  $-$ , by Art. 23. which makes it  $-d$ : then dividing  $ab$  by  $-a$ , the Quotient is  $b$ ; but as the Signs of  $ab$  and  $a$  are different, therefore by Art. 23. prefix the Sign  $-$  to  $b$ , and  $-d - b$  is the Quotient required.

*Exam. 3.* Dividing  $-dnz$  by  $dz$  the Quotient is  $-n$ , by Art. 20. and 23. and for the same Reason dividing  $-dzy$  by  $dz$ , the Quotient is  $-y$ , which placing after  $-n$ , we have  $-n - y$ , the Quotient required.

Divide	$maz + mzd$	$-dab - dby$	$-azx - azb$
By	$-mz$	$db$	$az$
Quotient	$-a - d$	$-a - y$	$-x - b$

The Truth of these Operations are likewise proved from multiplying the Quotient by the Divisor, and if it agrees with the Dividend in its Quantities and Signs, the Work is true, otherwise not.

25. *Case 3.* But when there are Co-efficients joined to the Quantities, divide the Co-efficients as in common Arithmetic; and to their Quotients join the Quotient of the Quantities found by the foregoing Directions; but cautiously remember, that if the Signs of the Quantities that are divided are alike, the Quotient must have the affirmative Sign, as at Art. 20. but if the Signs of the Quantities that are divided are unlike, then the Quotient must have the Sign  $-$  prefix to it, by Art. 23.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Divide	$16am$	$8yz$	$-24dm$	$-18ma$
By	$2a$	$2z$	$-6d$	$-6a$
Quotient	$8m$	$4y$	$4m$	$3m$

*Exam.*

*Exam. 1.* Dividing 16 by 2 the Quotient is 8, and  $a m$  divided by  $a$  the Quotient is  $m$ , joining 8 to the  $m$  it is  $8 m$ , and as  $16 a m$  and  $2 a$  have the same Sign, hence by Art. 20. the Sign  $+$  must be prefixt to  $8 m$ , therefore the Quotient is  $+ 8 m$  or  $8 m$ .

*Exam. 2.* Dividing 8 by 2 the Quotient is 4, and dividing  $y z$  by  $z$  the Quotient is  $y$ , joining the 4 to the  $y$  it is  $4 y$ ; but as  $8 y z$  and  $2 z$  have the same Sign, therefore by Art. 20. prefix the Sign  $+$  to  $4 y$ , hence  $+ 4 y$  or  $4 y$  is the Quotient required.

*Exam. 3.* Dividing 24 by 6 the Quotient is 4, and dividing  $d m$  by  $d$  the Quotient is  $m$ , joining 4 to the  $m$  it is  $4 m$ ; but as  $24 d m$  and  $6 d$  have the same Sign, prefix the Sign  $+$  to  $4 m$ , hence  $+ 4 m$  or  $4 m$  is the Quotient required.

*Exam. 4.* Dividing 18 by 6 the Quotient is 3, and dividing  $m a$  by  $a$  the Quotient is  $m$ , joining 3 to the  $m$  it is  $3 m$ , and as  $18 m a$  and  $6 a$  have the same Sign, therefore by Art. 20. the Quotient is  $+ 3 m$  or  $3 m$ .

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
Divide	$- 15 a y$	$- 8 d m$	$28 y z$	$- 12 d a$
By	$\underline{3 a}$	$\underline{- 4 d}$	$\underline{- 7 y}$	$\underline{3 a}$
Quotient	$- 5 y$	$2 m$	$- 4 z$	$- 4 d$

*Exam. 5.* Dividing 15 by 3 the Quotient is 5, and dividing  $a y$  by  $a$  the Quotient is  $y$ , joining 5 to the  $y$  it is  $5 y$ , but as the Signs of  $15 a y$  and  $3 a$  are different, therefore by Art. 23. prefix the Sign  $-$  to  $5 y$ , and  $- 5 y$  is the Quotient required.

*Exam. 6.* Dividing 8 by 4 the Quotient is 2, and dividing  $d m$  by  $d$  the Quotient is  $m$ , joining the 2 and  $m$  it is  $2 m$ ; but as  $8 d m$  and  $4 d$  have the same Sign, prefix the Sign  $+$  to  $2 m$ , by Art. 20. and  $+ 2 m$  or  $2 m$  is the Quotient required.

*Exam. 7.* Dividing 28 by 7 the Quotient is 4, and dividing  $y z$  by  $y$  the Quotient is  $z$ , joining the 4 and  $z$  it is  $4 z$ ; but as  $28 y z$  and  $7 y$  have different Signs, therefore by Art. 23. prefix the Sign  $-$  to  $4 z$ , so will  $- 4 z$  be the Quotient required.

*Exam. 8.* Dividing 12 by 3 the Quotient is 4, and dividing  $d a$  by  $a$  the Quotient is  $d$ , joining the 4 and  $d$  it is  $4 d$ ; but as the Signs of  $12 d a$  and  $3 a$  are different, therefore by Art. 23. prefix the Sign  $-$  to  $4 d$ , and then  $- 4 d$  is the Quotient required.

Divide

Divide	$— 32 a m$	$18 d z a$	$— 22 y m n$	$16 a z$
By	$— 8 m$	$9 a$	$11 y n$	$— 8 z$
Quotient	$4 a$	$2 d z$	$— 2 m$	$— 2 a$

26. And if there are two or more Quantities connected together with Co-efficients, to be divided by any single Quantity and its Co-efficient, the Operation is still performed in the same Manner, connecting the particular Quotients as at Art. 22, and 24. still carefully remembering that when the Quantities that are divided have like Signs, whether *affirmative* or *negative*, the Quotient must have the *affirmative* Sign; but if the Signs of the Quantities that are divided are unlike, then the Quotient must have the Sign  $—$  prefix to it.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
Divide	$4 a m + 12 a d$	$— 16 m y + 24 m z$	$28 d n — 21 d b$
By	$2 a$	$— 4 m$	$7 d$
Quotient	$2 m + 6 d$	$4 y — 6 z$	$4 n — 3 b$

*Exam. 1.* Dividing  $4 a m$  by  $2 a$  the Quotient is  $2 m$ , by Art. 25. and dividing  $12 a d$  by  $2 a$  the Quotient is  $6 d$ , and because the Signs of  $2 a$  and  $12 a d$  are alike, prefix the Sign  $+$  to  $6 d$ , and we have  $2 m + 6 d$ , the Quotient required.

*Exam. 2.* Dividing  $— 16 m y$  by  $— 4 m$  the Quotient is  $4 y$ , by Art. 25. for the Signs of  $16 m y$  and  $4 m$  are alike, and dividing  $24 m z$  by  $— 4 m$  the Quotient is  $— 6 z$ , for  $6 z$  must have the negative Sign prefix to it, the Signs of  $24 m z$  and  $4 m$  being unlike; hence  $4 y — 6 z$  is the Quotient required.

*Exam. 3.* Dividing  $28 d n$  by  $7 d$  the Quotient is  $4 n$ , or  $+ 4 n$ , for the Signs of  $28 d n$  and  $7 d$  are alike: and dividing  $— 21 d b$  by  $7 d$  the Quotient is  $— 3 b$ , for  $3 b$  must have the negative Sign prefix to it, as the Signs of  $21 d b$  and  $7 d$  are unlike, hence  $4 n — 3 b$  is the Quotient required.

Divide	$16 p a — 28 p d$	$— 24 n m + 36 m z$	$16 z u — 4 z d$
By	$— 4 p$	$— 4 m$	$2 z$
Quotient	$— 4 a + 7 d$	$6 n — 9 z$	$8 u — 2 d$

The Truth of these Operations are proved likewise from multiplying the Quotient by the Divisor, for if the Work is true, the Product will agree with the Dividend in its Quantities and Signs: In the last Example the Divisor is  $2 z$ , and the Quotient is  $8 u — 2 d$ , now if we

Multiply

$$\begin{array}{r} \text{Multiply } 8u - 2d \\ \text{By } 2z \\ \hline \end{array}$$

$16zu - 4zd$  the Product is the same as the given Dividend, and so may the other Examples be proved.

27. *Case 4.* But when any Quantities in the Dividend are not the same with those in the Divisor, then place down the Dividend with its Signs and Co-efficients, under which drawing a Line, and after the Manner of Vulgar Fractions place the Divisor with its Signs and Co-efficients, and this will be the Quotient required.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Divide	$b$	$am$	$3my$	$2dy$
By	$a$	$d$	$z$	$b$
Quotient	$\frac{b}{a}$	$\frac{am}{d}$	$\frac{3my}{z}$	$\frac{2dy}{b}$

*Exam. 1.* Because  $b$  and  $a$  are different Quantities, place down the Dividend  $b$ , under which draw a Line, and place the Divisor  $a$ , so is  $\frac{b}{a}$  the Quotient required.

*Exam. 2.* Because  $am$  and  $d$  are different Quantities, place down  $am$  the Dividend, draw a Line under it, and place the Divisor  $d$ , so is  $\frac{am}{d}$  the Quotient required.

*Exam. 3.* Because  $3my$  and  $z$  are different Quantities, place down  $3my$  the Dividend, under it draw a Line, and place  $z$  the Divisor, so is  $\frac{3my}{z}$  the Quotient required.

*Exam. 4.* Because  $2dy$  and  $b$  are different Quantities, place down  $2dy$  the Dividend, draw a Line under it, and place  $b$  the Divisor, and  $\frac{2dy}{b}$  is the Quotient required.

Divide	$2ma$	$5dz$	$21ma$	$8yd$
By	$3y$	$2y$	$5d$	$3z$
Quotient	$\frac{2ma}{3y}$	$\frac{5dz}{2y}$	$\frac{21ma}{5d}$	$\frac{8yd}{3z}$



# D I V I S I O N.

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Divide	$ma$	$7d$	$3mbc$	$24d$
By	$7y$	$mz$	$yd$	$7yz$
Quotient	$\frac{ma}{7y}$	$\frac{7d}{mz}$	$\frac{3mbc}{yd}$	$\frac{24d}{7yz}$

28. When two or more Quantities connected by the Signs  $+$  or  $-$  are to be divided by any single Quantity, and the Quantities in the Dividend are different from those in the Divisor, then having set down all the Quantities in the Dividend with their Signs and Co-efficients, draw a Line under them all, under which place the Divisor as before, and this will be the Quotient required.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
Divide	$2a + 3b$	$7y - 2m$	$15z - 7da$
By	$5m$	$3n$	$4y$
Quotient	$\frac{2a + 3b}{5m}$	$\frac{7y - 2m}{3n}$	$\frac{15z - 7da}{4y}$

*Exam. 1.* Because  $2a + 3b$  the Dividend and  $5m$  the Divisor are different, therefore place down  $2a + 3b$ , under which draw a Line, and place the Divisor  $5m$ , so is  $\frac{2a + 3b}{5m}$  the Quotient required.

*Exam. 2.* Because  $7y - 2m$  the Dividend and  $3n$  the Divisor are different, therefore place down  $7y - 2m$ , under which draw a Line, and place the Divisor  $3n$ , so is  $\frac{7y - 2m}{3n}$  the Quotient required.

*Exam. 3.* Because  $15z - 7da$  the Dividend and  $4y$  the Divisor are different, therefore place down  $15z - 7da$  the Dividend, under which draw a Line, and place  $4y$  the Divisor, so is  $\frac{15z - 7da}{4y}$  the Quotient required.

Divide	$4ma - 3d$	$7db - 5xz$	$19m - 15p$
By	$5z$	$3y$	$7y$
Quotient	$\frac{4ma - 3d}{5z}$	$\frac{7db - 5xz}{3y}$	$\frac{19m - 15p}{7y}$

G

Divide

$$\begin{array}{r}
 \text{Divide} \quad 3dx - 5b \quad 7ym - 3dn \quad 25dp + 5yz - 17m \\
 \text{By} \quad 2y \quad 5u \quad 7a \\
 \hline
 \text{Quotient} \quad \frac{3dx - 5b}{2y} \quad \frac{7ym - 3dn}{5u} \quad \frac{25dp + 5yz - 17m}{7a}
 \end{array}$$

29. If there are two or more Quantities connected by the Signs  $+$  or  $-$ , to be divided by two or more Quantities connected by the Signs  $+$  or  $-$ , but the Quantities in the Dividend are different from those in the Divisor, it is only placing down the Dividend as before, under which drawing a Line, and place in like Manner the Divisor, and this will be the Quotient required.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
Divide	$2a + m$	$5y - 7d$	$-14m + 5z - 11x$
By	$5d + 3y$	$3a + 2m$	$3y - 2d$
Quotient	$\frac{2a + m}{5d + 3y}$	$\frac{5y - 7d}{3a + 2m}$	$\frac{-14m + 5z - 11x}{3y - 2d}$

*Exam. 1.* Because the Quantities in the Dividend and Divisor are unlike, therefore place down  $2a + m$  the Dividend with its Co-efficients and Signs, under which draw a Line, and place  $5d + 3y$  the Divisor, so is  $\frac{2a + m}{5d + 3y}$  the Quotient required.

*Exam. 2.* Because  $5y - 7d$  the Dividend is different from  $3a + 2m$  the Divisor, therefore place down  $5y - 7d$  the Dividend, under which draw a Line, and place  $3a + 2m$  the Divisor, so is  $\frac{5y - 7d}{3a + 2m}$  the Quotient required.

*Exam. 3.* Because  $-14m + 5z - 11x$  the Dividend is different from  $3y - 2d$  the Divisor, therefore place down  $-14m + 5z - 11x$  the Dividend, draw a Line under it, and place  $3y - 2d$  the Divisor, and  $\frac{-14m + 5z - 11x}{3y - 2d}$  is the Quotient required.

$$\begin{array}{r}
 \text{Divide} \quad 4m - 5y \quad -21pm + 19zy \quad 14yz - 9dx \\
 \text{By} \quad 3x + 2z \quad 5d - 2b \quad -3m + 5n \\
 \hline
 \text{Quotient} \quad \frac{4m - 5y}{3x + 2z} \quad \frac{-21pm + 19zy}{5d - 2b} \quad \frac{14yz - 9dx}{-3m + 5n}
 \end{array}$$

Divide

Divide	$-4a + 5m - 3d$	$4a + 3y - 5x$	$2a + 3y$
By	$7z - 8y$	$-7d + 11m$	$-5z - 7n$
Quotient	$-4a + 5m - 3d$	$4a + 3y - 5x$	$2a + 3y$
	$7z - 8y$	$-7d + 11m$	$-5z - 7n$

30. It may be just observed to the Learner, that when any Quantity is divided by itself, or the Dividend and Divisor are alike, that then the Quotient will be *Unity*, or 1. And if the Signs of the Quantities to be divided are alike, the Quotient must have the Sign +; but if the Signs of the Quantities to be divided are unlike, then to the Quotient, or 1, prefix the Sign —.

Divide	$2ab$	$14mn$	$-5dz$	$+7y$
By	$2ab$	$14mn$	$-5dz$	$-7y$
Quotient	1	1	1	1

For by Art. 25. if we divide the Co-efficients, the Quotient will be *Unity*, or 1; then, by Art. 20: rejecting all those Quantities that are alike, both in the Dividend and Divisor, the Quantities all vanish, and there will be none to be joined to the *Unity*, or 1; whence, in such Cases as these, *Unity*, or 1, is the Quotient required.

It may be further observed, that if an absolute or pure Number is the Divisor, the Co-efficients in the Dividend, if there are more than one, must be divided by the Divisor, and to each of these Quotients join the respective Quantities of the Dividend, as at Art. 26.

Divide	$24ma + 18yz$	$16za + 24ym$	$+ 14yd + 35z$
By	6	8	7
Quotient	$4ma + 3yz$	$2za + 3ym$	$- 2yd - 5z$

But if the Divisor will not exactly divide the Co-efficients of the Dividend, then place the Dividend and Divisor in the Manner of *Vulgar Fractions*, as in the foregoing Articles.

The Method of dividing compound Quantities by one another, where the Operation is continued as in common Arithmetic, being generally perplexing to Learners, it will be explained in the Method of solving *Quadratic Equations*, this Division not being necessary in the present Design before we come to that Part of the Work.

# INVOLUTION.

31. **T**HIS is only raising of Powers from any given Root, and therefore is performed by Multiplication: For the Quantity which is given being multiplied by itself will be the Square of that Quantity, that Product being multiplied by the given Quantity, this Product will be the Cube of that Quantity, and that Product multiplied again by the given Quantity, will be the fourth Power of that Quantity; and so on as in common Arithmetick.

To find the Cube of  $a$

The Square of  $a$   $\frac{a}{a \ a}$

The Cube of  $a$   $\frac{a}{a \ a \ a}$

To find the Cube of  $b$

The Square of  $b$   $\frac{b}{b \ b}$

The Cube of  $b$   $\frac{b}{b \ b \ b}$

To find the Cube of  $2y$

Now  $2y$  multiplied by  $2y$  the }  
Product will be by Art. 13. }

$2y$   
 $2y$   
 $4yy$  the Square of  $2y$

And  $4yy$  multiplied by  $2y$ , the }  
Product will be by Art. 13. }

$2y$   
 $8yyy$  the Cube of  $2y$

To find the Cube of  $3z$

Now  $3z$  multiplied by  $3z$ , the }  
Product will be by Art. 13. }

$3z$   
 $3z$   
 $9zz$  the Square of  $3z$

And  $9zz$  multiplied by  $3z$ , the }  
Product will be by Art. 13. }

$3z$   
 $27zzz$  the Cube of  $3z$

To find the 4th Power of  $-2x$

$$\begin{array}{r} -2x \\ \hline \end{array}$$

Now  $-2x$  multiplied by  $-2x$ , the Product is by Art. 9. and 13.

$$\begin{array}{r} 4xx \text{ the Square of } -2x \\ \hline -2x \end{array}$$

And  $4xx$  multiplied by  $-2x$ , the Product is by Art. 13. and 16.

$$\begin{array}{r} -8xxx \text{ the Cube of } -2x \\ \hline -2x \end{array}$$

And  $-8xxx$  multiplied by  $-2x$ , the Product is by Art. 9. and 13.

$$\begin{array}{r} 16xxxx \text{ the 4th Power of } -2x. \\ \hline \end{array}$$

In like Manner any other single Quantity may be raised to any required Power, and if in the given Quantity there are more Letters than one, it is done in the same Manner.

To find the 4th Power of  $2ab$

$$\begin{array}{r} 2ab \\ \hline 4aabb \text{ the Square of } 2ab \\ \hline 2ab \\ \hline 8aaabbb \text{ the Cube of } 2ab \\ \hline 2ab \\ \hline 16aaaaabbbb \text{ the 4th Power of } 2ab. \end{array}$$

32. If there are two or more Quantities connected by the Signs  $+$  or  $-$ , to be raised to any given Power, it is still performed by common Multiplication. Two Quantities when connected by the Sign  $+$ , is commonly called a *Binomial*.

To raise the *Binomial*,

or  $a + b$  to the third Power or Cube.

$$\begin{array}{r} a + b \\ \hline aa + ab \text{ the Prod. of } a + b \text{ multip. by } a, \text{ by Art. 10.} \\ ab + bb \text{ the Prod. of } a + b \text{ multip. by } b, \text{ by Art. 10.} \\ \hline aa + 2ab + bb \text{ the Sum of these two Products, or} \\ a + b \text{ the Square of } a + b. \\ \hline aaa + 2aab + abb \text{ the Product of } aa + 2ab + bb \\ \text{multiplied by } a, \text{ by Art. 10.} \\ aab + 2abb + bbb \text{ the Product of } aa + 2ab + bb \\ \text{multiplied by } b, \text{ by Art. 10.} \\ \hline aaa + 3aab + 3abb + bbb \text{ the Sum of these two Pro-} \\ \text{ducts, or the Cube of } a + b. \end{array}$$

When

When two Quantities are connected by the Sign —, it is commonly called a *Residual*.

To raise the *Residual*,

or  $x - y$  to the third Power or Cube.

$$\begin{array}{l}
 \underline{x - y} \\
 xx - xy \text{ the Product of } x - y \text{ multiplied by } x. \\
 -xy + yy \text{ the Product of } x - y \text{ multiplied by } -y. \\
 \underline{xx - 2xy + yy} \text{ the Sum of these two Products, or} \\
 \text{the Square of } x - y. \\
 \underline{x - y} \\
 \underline{xx - 2xy + yy} \text{ the Product of } xx - 2xy + yy \\
 \text{multiplied by } x. \\
 -xxy + 2xyy - yyy \text{ the Product of } xx - 2xy + yy \\
 \text{multiplied by } -y. \\
 \underline{xx - 3xxy + 3xyy - yyy} \text{ the Sum of these two Pro-} \\
 \text{ducts, or the Cube of } x - y.
 \end{array}$$

And if these compound Quantities have Co-efficients, the Work still proceeds as at Art. 18.

To raise the *Binomial*,

or  $2a + 3b$  to the third Power.

$$\begin{array}{l}
 \underline{2a + 3b} \\
 4aa + 6ab \text{ the Product of } 2a + 3b \text{ multiplied by } 2a. \\
 6ab + 9bb \text{ the Product of } 2a + 3b \text{ multiplied by } 3b. \\
 \underline{4aa + 12ab + 9bb} \text{ the Sum of these two Products,} \\
 \text{or the Square of } 2a + 3b. \\
 \underline{2a + 3b} \\
 8aaa + 24aab + 18abb \text{ the Product of } 4aa + 12ab \\
 + 9bb \text{ multiplied by } 2a. \\
 12aab + 36abb + 27bbb \text{ the Product of } 4aa + 12ab \\
 + 9bb \text{ multiplied by } 3b. \\
 \underline{8aaa + 36aab + 54abb + 27bbb} \text{ the Sum of these two} \\
 \text{Products, or the Cube of } 2a + 3b.
 \end{array}$$

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To raise  $3m + 2y$  to the third Power.

$$\begin{array}{r}
 3m + 2y \\
 \hline
 9mm + 6my \\
 6m^2 + 4yy \\
 \hline
 9mm + 12my + 4yy \text{ the Square of } 3m + 2y \\
 3m + 2y \\
 \hline
 27mmm + 36mmy + 12myy \\
 18mmy + 24myy + 8yyy \\
 \hline
 27mmm + 54mmy + 36myy + 8yyy \text{ the Cube of } 3m + 2y.
 \end{array}$$

To raise  $a - 2b$  to the third Power.

$$\begin{array}{r}
 a - 2b \\
 \hline
 aa - 2ab \\
 - 2ab + 4bb \\
 \hline
 aa - 4ab + 4bb \text{ the Square of } a - 2b \\
 a - 2b \\
 \hline
 aaa - 4aab + 4abb \\
 - 2aab + 8abb - 8bbb \\
 \hline
 aaa - 6aab + 12abb - 8bbb \text{ the Cube of } a - 2b.
 \end{array}$$

In this Example I have placed the same Quantities under each other, for the more commodious adding them, though this is not necessary, and is a Knowledge the Learner will acquire from his own Observation.

# EVOLUTION.

33. **T**HIS is the Extraction of Roots, and therefore opposite to Involution, and as Equations in which the unknown Quantity rises above the Square are generally adfected, and resolved by the Method of *Converging Series*, we shall consider the Square Root only; and give such Directions that the Learner may generally know, whether the Square Root of such Quantities as commonly occur in the Solution of Questions can be extracted or not.

Now so many Times as any Letter is repeated, so high is the Power of that Letter said to be. Thus,  $a$  is  $a$  to the first Power:  $aa$  is  $a$  to the second Power or Square: and  $aaaa$  is  $a$  to the fourth Power, &c. as in Involution.

And to extract the Root of any simple Quantity, consider how many Times the Letter is repeated, or how high the Power of it is, and if it appears to be the second, third, fourth, or any other Power, divide that Figure which expresses the Height of the Power by 2, and if it does not divide it exactly, it is a *Surd* Quantity, and has no Square Root; but if it divides it exactly, set down the Quantity whose Root you are extracting as many Times as the Quotient of the above Division directs, and that will be the Square Root required.

*Exam. 1. Exam. 2. Exam. 3.*

To extract the Square Root of	$aa$	$bbbb$	$bbbbbb$
The Square Root is	$a$	$bb$	$bbb$

*Exam. 1.* Here  $a$  is repeated twice, or to the second Power; now dividing 2 by 2 the Quotient is 1, therefore setting down  $a$  once, or  $a$ , it is the Square Root required.

*Exam. 2.* Here  $b$  is repeated four times, or to the fourth Power; now dividing 4 by 2 the Quotient is 2, therefore setting down  $b$  twice, or  $bb$ , it is the Square Root required.

*Exam. 3.* Here  $b$  is repeated six times, or to the sixth Power; now dividing 6 by 2 the Quotient is 3, therefore setting down  $b$  three times, or  $bbb$ , it is the Square Root required.

The Truth of these Operations are proved by Multiplication, for if the Work is right, the Square Root being multiplied by itself will produce the Quantity from which the Root was extracted. Thus in Example 2,

The Square Root is	$bb$
Which being multiplied by itself	$bb$
The Product is the given Square	$bbbb$

And so of any other Example.

*Exam. 4.*

*Exam. 5.*

To extract the Square Root of	$aaaa$	$dddddd$
The Square Root is	$aa$	$ddd$

*Exam. 4.* Here  $a$  is repeated four times, or to the fourth Power; now dividing 4 by 2 the Quotient is 2, which shows that  $a$  must be repeated twice, that is,  $aa$  is the Square Root required.

*Exam.*



*Exam. 5.* Here  $d$  is repeated six times, or to the sixth Power; now dividing 6 by 2 the Quotient is 3, which shows that  $d$  must be repeated three times, and consequently  $ddd$  is the Square Root required.

And if the Quantity, whose Root is to be extracted, has different Letters, then consider if the Number of Times each Letter is repeated can be divided by 2 without any Remainder, and if they can, set down each Letter so many Times as the Quotient of the respective Division directs, and joining them, this will be the Square Root required; but if the Number of Times any one Letter is repeated cannot be divided by 2, then the whole Quantity has no Square Root.

*Exam. 1.   Exam. 2.   Exam. 3.*

To extract the Square Root of	$aabbbb$	$aaaadddd$	$mmp p$
The Square Root is	$abb$	$aadd$	$mp$

*Exam. 1.* Here  $a$  is repeated twice, and 2 being divided by 2 the Quotient is 1, which shows  $a$  must be taken only once, or  $a$ . Now  $b$  is repeated four Times, or to the fourth Power, and 4 being divided by 2 the Quotient is 2, which shows  $b$  must be repeated twice, or  $bb$ , now joining  $a$  to  $bb$ , and  $abb$  is the Square Root required.

*Exam. 2.* Here  $a$  is repeated to the fourth Power, and dividing 4 by 2 the Quotient is 2, which shows that  $a$  must be repeated twice, that is, it must be  $aa$ : Again,  $d$  is repeated to the fourth Power, and dividing 4 by 2, the Quotient is 2, which shows  $d$  must be repeated to the second Power, or  $dd$ . Now joining  $aa$  to  $dd$ , we have  $aadd$  for the Square Root required.

By the same Method of reasoning we shall find in Example 3, that the Square Root of  $mmp p$  is  $mp$ .

But when it is found that the given Quantity has not such a Root as is required, then the Square Root of it is expressed by prefixing this Sign  $\sqrt{\phantom{x}}$  before it.

*Exam. 1.   Exam. 2.   Exam. 3.*

Required the Square Root of	$a$	$bbb$	$dddd$
The Square Root is	$\sqrt{a}$	$\sqrt{bbb}$	$\sqrt{dddd}$

*Exam. 1.* Because  $a$  is only repeated once, and as we cannot divide 1 by 2, and have the Quotient a whole Number, therefore I conclude  $a$  is a *Surd* Quantity, and accordingly, to express the

Square Root of  $a$ , prefix the Sign  $\sqrt{\phantom{x}}$  to it, so is  $\sqrt{a}$  the Square Root required.

*Exam. 2.* Here  $b$  is repeated three times, and because 3 cannot be divided by 2, and have no Remainder, therefore I conclude  $bbb$  is a *Surd* Quantity, and to express the Square Root of it, prefix the Sign  $\sqrt{\phantom{x}}$  to it, so is  $\sqrt{bbb}$  the Square Root required.

*Exam. 3.* Here  $d$  being repeated five times, and as we cannot divide 5 by 2, and have no Remainder, therefore I conclude that  $dddd$  is a *Surd* Quantity, and to express the Square Root of it, prefix the Sign  $\sqrt{\phantom{x}}$  to it, so is  $\sqrt{dddd}$  the Square Root required.

34. But to extract the Square Root of compound Quantities, or those connected by the Signs  $+$  or  $-$ , observe,

*First*, There must be three Quantities to make it a Square, for  $a + b$  multiplied by itself, or squared, the Product is  $aa + 2ab + bb$ , by Art. 32. whence if there are only two Quantities it is a *Surd*. I take no Notice of any greater Number of Quantities than three, which may compose a Square, as they seldom occur in any Operation.

*Secondly*, Whether these three Quantities have two different Letters only; there may be Cases in which there are more than two different Letters in these three Quantities, but as they seldom happen, I choose not to perplex the Learner with them.

*Thirdly*, If two of these three Quantities are pure Powers of those two Letters; that is, in the Square of  $a + b$  there is  $aa$  and  $bb$ , pure Powers of the Quantities  $a$  and  $b$ .

*Fourthly*, Whether both these pure Powers of the two different Letters have the Sign  $+$  before them.

*Fifthly*, If the third of the above three Quantities is twice the Product of the Square Root of the two pure Powers of the two different Letters, that is, the Square of  $a + b$  being  $aa + 2ab + bb$ , the Quantity  $2ab$  is twice the Product of the Square Root of  $aa$  and  $bb$ ; and this Quantity may have either the Sign  $+$  or  $-$ .

Now if the given Quantity, whose Root is to be extracted, answers these Particulars, its Square Root may be extracted thus.

*Sixthly*, Extract the Square Root of the two pure Powers of the two different Letters, according to the Directions at Art. 33.

*Seventhly*, If the Quantity mentioned at the *fifth* Particular has the *negative* Sign, connect the two Roots mentioned in the last Particular with the Sign  $-$ , and it will be the Square Root required.

*Eighthly*,

*Eighthly*, But if the Quantity mentioned at the *fifth* Particular has the Sign  $+$ , then connect those two Roots with the Sign  $+$ , and this will be the Square Root required.

Now let it be required to extract the Square Root of  $aa + 2ab + bb$ .

Here are three Quantities by the *first* Particular.

They have likewise two different Letters, *viz.*  $a$  and  $b$ , by the *second* Particular.

Two of these Quantities, *viz.*  $aa$  and  $bb$ , are pure Powers of the two Letters  $a$  and  $b$ , by the *third* Particular.

And both these pure Powers, *viz.*  $aa$  and  $bb$ , have the Sign  $+$ , by the *fourth* Particular.

Now suppose we neglect the Consideration of the *fifth* Particular, and attempt the Extraction of the Root by the *sixth* Particular.

Then the Square Root of  $aa$ , is by Art. 33.  $\text{---} \quad a$

And the Square Root of  $bb$ , is by the same  $\text{---} \quad b$

And now the third Quantity  $2ab$  being twice the Product of the Roots  $a$  and  $b$ , and having the Sign  $+$ ,

Therefore by the *eighth* Particular, I connect  $a$  and  $b$  with the Sign  $+$ , then it is  $\text{---} \quad a + b$

Hence I suppose  $a + b$  to be the Square Root of  $aa + 2ab + bb$ .

But to prove the Truth of the Operation, multiply the Root by itself, and if the Product agrees with the given Quantity, in its Quantities, Signs, and Co-efficients, the Work is right; if not the Work is either erroneous, or has no Square Root, and is a *Surd* Quantity.

The Root of the last Example }  
was supposed to be

Which multiplied by itself

$$\begin{array}{r} a + b \\ a + b \\ \hline aa + ab \\ ab + bb \\ \hline aa + 2ab + bb \end{array}$$

The Product is the given Quantity, which proves that  $a + b$  is the Square Root of  $aa + 2ab + bb$ .

Required the Square Root of  $aa + 2za + zz$ .

Here are three Quantities by the *first* Particular.

They have likewise two different Letters,  $a$  and  $z$ , by the *second* Particular.

Two of these Quantities, *viz.*  $aa$  and  $zz$ , are pure Powers of  $a$  and  $z$ , by the *third* Particular, whose Square Roots are  $a$  and  $z$ .

And both these Powers have the Sign  $+$  by the *fourth* Particular.

Now the third Quantity  $2za$  is twice the Product of  $a$  and  $z$ , the Square Roots of the two pure Powers  $aa$  and  $zz$ .

Then to extract the Square Root of  $aa + 2za + zz$  by the *sixth* Particular.

The Square Root of  $aa$  by Art. 33. is  $\underline{\hspace{1cm}} \quad a$

The Square Root of  $zz$  by the same is  $\underline{\hspace{1cm}} \quad z$

Because the third Quantity  $2az$  has the Sign  $+$ , therefore by the *eighth* Particular, connect  $a$  and  $z$  with the Sign  $+$ , and  $a + z$  is the Square Root required.

To try if the Square Root is  
Multiply it by itself

$$\begin{array}{r} a + z \\ a + z \\ \hline aa + az \\ az + zz \\ \hline aa + 2az + zz \end{array}$$

The Product  $aa + 2az + zz$ , agreeing with the given Quantity, in the Quantities, Signs, and Co-efficients, it appears that  $a + z$  is the Square Root required.

To extract the Square root of  $mm - 2mp + pp$ .

Here are three Quantities by the *first* Particular.

They have likewise two different Letters  $m$  and  $p$ , by the *second* Particular.

Two of these three Quantities, *viz.*  $mm$  and  $pp$  are pure Powers of  $m$  and  $p$ , by the *third* Particular.

And both these Powers have the Sign  $+$ , by the *fourth* Particular.

Likewise the third Quantity  $-2mp$  is twice the Product of  $m$  and  $p$ , the Square Roots of the two pure Powers  $mm$  and  $pp$ .

Then according to the *sixth* Particular, the Square Root of  $mm$  is  $\underline{\hspace{1cm}} \quad m$

By the same, the Square Root of  $pp$  is  $\underline{\hspace{1cm}} \quad p$

But as the third Quantity  $2mp$  has the Sign  $-$ , therefore by the *seventh* Particular connect  $m$  and  $p$  with the Sign  $-$ , and  $m - p$  is the Square Root required.

To try if the Square Root is  
Multiply it by itself

$$\begin{array}{r} m - p \\ m - p \\ \hline mm - mp \\ -mp + pp \\ \hline mm - 2mp + pp \end{array}$$

The Product  $mm - 2mp + pp$ , agreeing in every thing with the given Quantity, it proves  $m - p$  is the Square Root required.

By

By the same Method of reasoning it will be found that the Square Root of  $xx + 2xz + zz$ , is  $x + z$ .

And that the Square Root of  $aa - 2ad + dd$ , is  $a - d$ .

And that the Square Root of  $xx - 2xm + mm$ , is  $x - m$ .

And if it was required to extract the Square Root of  $aa + \frac{bb}{4}$ , in extracting the Root of the Fractional Quantity extract the Root of the Numerator for a new Numerator, and the Root of the Denominator for a new Denominator.

Here the two pure Powers are  $aa$  and  $\frac{bb}{4}$ .

But the Square Root of  $aa$  is  $a$

And the Square Root of  $\frac{bb}{4}$  is  $\frac{b}{2}$

And connecting these we have  $a + \frac{b}{2}$

The Square Root required.

To prove the Truth of this Operation, multiply  $a + \frac{b}{2}$  by itself,

$$\begin{array}{r} a + \frac{b}{2} \\ a + \frac{b}{2} \\ \hline aa + \frac{ab}{2} \\ \frac{ab}{2} + \frac{bb}{4} \\ \hline aa + ab + \frac{bb}{4} \end{array}$$

$a$  multiplied by  $a$  the Product is  $aa$ , and  $\frac{b}{2}$  multiplied by  $a$  is  $\frac{ab}{2}$ , (for making  $a$  an improper Fraction  $\frac{a}{1}$  as in common Arithmetic, and multiplying the two Numerators  $a$  and  $b$  for a new Numerator, and the two Denominators  $1$  and  $2$  for a new Denominator, we have  $\frac{ab}{2}$ ) and  $\frac{b}{2}$  multiplied by  $\frac{b}{2}$  produces  $\frac{bb}{4}$  by the same Rule; and in the Products the Fractions  $\frac{ab}{2}$  and  $\frac{ab}{2}$  having the same Denominator, adding them accord-

ing

ing to the Rule for Addition of Vulgar Fractions in Arithmetic, the Sum is  $\frac{2ab}{2}$ , but rejecting the 2 by the Rule for

Division of *Algebra* the Sum is  $ab$ .

Therefore when any one of the Quantities appears in a *Fractional* Manner, we must extract the Square Root of both the Numerator and Denominator, placing the Square Root of the Numerator for a new Numerator, and the Square Root of the Denominator for a new Denominator, and try the Work as before.

But if we cannot extract the Square Root of both the Numerator and Denominator, then we conclude the given Quantity to be a Surd.

Now by this Reasoning we shall find the Square Root of  $xx + xa + \frac{aa}{4}$ , to be  $x + \frac{a}{2}$ .

And that the Square Root of  $mm - my + \frac{yy}{4}$ , is  $m - \frac{y}{2}$ .

Suppose it was required to extract the Square Root of  $xx + 2xn - nn$ .

Here are three Quantities by the *first* Particular.

They have likewise two different Letters,  $x$  and  $n$ , by the *second* Particular.

Two of these three Quantities, *viz.*  $xx$  and  $nn$ , are pure Powers of  $x$  and  $n$ .

But both these Powers have not the Sign  $+$ , for it is  $-nn$ , therefore by the *fourth* Particular, I conclude that the given Quantity  $xx + 2xn - nn$  is a *Surd* Quantity, and its Square Root cannot be extracted any otherwise than by prefixing the Sign  $\sqrt{\phantom{x}}$  to it, as in Art. 33. Thus,  $\sqrt{xx + 2xn - nn}$  is, or expresses the Square Root of  $xx + 2xn - nn$ .

Let it be required to extract the Square Root of  $aa + 5ab + bb$ .

Here are three given Quantities by the *first* Particular.

They have likewise two different Letters,  $a$  and  $b$ , by the *second* Particular.

Two of these Quantities, *viz.*  $aa$  and  $bb$ , are pure Powers of  $a$  and  $b$ .

And both these Powers have the Sign  $+$  by the *fourth* Particular.

But then the third Quantity  $5ab$  is not twice the Product of the Square Roots of  $aa$  and  $bb$ ; for their Roots being  $a$  and  $b$ , if they are multiplied the Product is  $ab$ , and that being multiplied by 2 it is  $2ab$ : Whereas the third Quantity in the given

Example

Example is  $5ab$ . Hence, I conclude that  $aa + 5ab + bb$  is a *Surd* Quantity, and to express its Square Root I prefix to it the Sign  $\sqrt{\phantom{x}}$ , so will  $\sqrt{aa + 5ab + bb}$  be the Square Root of  $aa + 5ab + bb$ .

And if it was required to extract the Square Root of  $aa + 2ab + \frac{bb}{5}$ , it will be found a *Surd* Quantity, it being impossible to extract the Square Root of  $5$ , therefore prefix the Sign  $\sqrt{\phantom{x}}$  to  $aa + 2ab + \frac{bb}{5}$ , and then  $\sqrt{aa + 2ab + \frac{bb}{5}}$  is the Square Root required.

For the same Reason the Square Root of  $xx + 2xa + \frac{aa}{3}$  is  $\sqrt{xx + 2xa + \frac{aa}{3}}$ , it being impossible to extract the Square Root of  $3$ .

When the Radical Sign or  $\sqrt{\phantom{x}}$  is to be prefixt to the Whole of any compound Quantity, draw the Top of the Sign over all those Quantities, which shews that they are all included under that Sign; for if the Sign was not to be drawn over all of them, it may be thought the Square Root of that Quantity was only to be extracted which stands next the Radical Sign.

To extract the Square Root of  $aaaa + 2aab + bb$ .

Here are three given Quantities by the *first* Particular.

They have likewise two different Letters,  $a$  and  $b$ , by the *second* Particular.

Two of these Quantities, *viz.*  $aaaa$  and  $bb$ , are pure Powers of  $a$  and  $b$ , by the *third* Particular.

And both these Powers have the Sign  $+$ , by the *fourth* Particular.

And the third Quantity  $2aab$  is twice the Product of the Square Roots of  $aaaa$  and  $bb$ .

For by the *sixth* Particular, the Square Root of  $aaaa$  is  $aa$

And by the same, the Square Root of  $bb$  is  $b$

And as the third Term  $2aab$  in the given Quantity has the Sign  $+$ , by the *eighth* Particular connect  $aa$  and  $b$ , the two Roots of  $aaaa$  and  $bb$ , with the Sign  $+$ , so is  $aa + b$  the Square Root of  $aaaa + 2aab + bb$ .

To prove which put down the }  
 supposed Square Root }  
 Which multiplied by itself

$$\begin{array}{r}
 aa + b \\
 aa + b \\
 \hline
 aaaa + aab \\
 aab - bb \\
 \hline
 aaaa + 2aab + bb
 \end{array}$$

The Product  $aaaa + 2aab + bb$ , agreeing with the given Quantity in every Particular, proves the Square Root to be as above.

To extract the Square Root of  $yyyy - 2yyx + xx$ .

Here the given Quantities agree with the first five Particulars as before.

By the *sixth* Particular I find the Square Root of  $yyyy$  is  $yy$

By the same, that the Square Root of  $xx$  is  $x$

But as the third Term  $-2yyx$  in the given Quantity has the Sign  $-$ , therefore by the *seventh* Particular I connect  $yy$  and  $x$  the two Roots with the Sign  $-$ , and say, or suppose  $yy - x$  to be the Square Root required.

To prove which put down }  
 the supposed Root }  
 Which multiply by itself

$$\begin{array}{r}
 yy - x \\
 yy - x \\
 \hline
 yyyy - yyx \\
 -yyx + xx \\
 \hline
 \end{array}$$

The Product, agreeing with }  
 the given Quantity }

$$yyyy - 2yyx + xx$$

By the same Method we shall find the Square Root of  $nnnn + 2nnd + dd$  to be  $nn + d$ .

And that the Square Root of  $xxxx + 2xxyy + yyyy$  is  $xx + yy$ .

And we shall find that  $dddd + 3ddy + yy$  is a *Surd* Quantity, and its Square Root must be expressed by prefixing the Radical Sign to it, thus  $\sqrt{dddd + 3ddy + yy}$ .

We shall likewise find that  $-pppp + 2pppy + yy$  is a *Surd* Quantity, and to extract its Square Root, is only to prefix to it the Radical Sign, thus  $\sqrt{-pppp + 2pppy + yy}$ .



## OF SURD QUANTITIES.

THESE are such Quantities whose Roots cannot be exactly extracted, and as they arise in the Resolution of Algebraic Questions, we shall explain so much of them only, as is necessary to the present Design.

*Addition of Surd Quantities, in which there are two Cases.*

35. *Case 1.* When the Quantities under the Radical Signs are alike, add the rational Quantities, or those which are without the Radical Sign together, by the Rules of Addition at *Art.* 1, 2, 3, 4, 5, 6, and to this join the *Surd* Quantities, and this will be the Sum required.

And if there appears to be no rational Quantities without the Radical Sign, then *Unity*, or 1, is always supposed to be the rational Quantity.

<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
To $\sqrt{am}$	$2\sqrt{dy}$	$6m\sqrt{d+a}$	$5y\sqrt{dm+z}$
Add $\sqrt{am}$	$\sqrt{dy}$	$4m\sqrt{d+a}$	$y\sqrt{dm+z}$
Sum $2\sqrt{am}$	$3\sqrt{dy}$	$10m\sqrt{d+a}$	$6y\sqrt{dm+z}$

*Exam. 1.* There being no rational Quantities, therefore *Unity*, or 1, is the rational Quantity to each. Now 1 added to 1 makes 2, to which joining the *Surd*  $\sqrt{am}$ , we have  $2\sqrt{am}$ , the Sum required.

*Exam. 2.* The rational Quantities being 2 and 1, their Sum is 3, to which joining  $\sqrt{dy}$  we have  $3\sqrt{dy}$ , the Sum required.

*Exam. 3.* The rational Quantities are 6 *m* and 4 *m*, which being added make 10 *m*, to which joining the *Surd*  $\sqrt{d+a}$  we have  $10m\sqrt{d+a}$ , the Sum required.

*Exam. 4.* The rational Quantities are 5 *y* and *y*, which being added make 6 *y*, to which joining the *Surd*  $\sqrt{dm+z}$  we have  $6y\sqrt{dm+z}$ , the Sum required.

<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>
To $13yd\sqrt{z-x}$	$15z\sqrt{da+p}$	$-7m\sqrt{da-y}$
Add $5yd\sqrt{z-x}$	$-3z\sqrt{da+p}$	$-2m\sqrt{da-y}$
Sum $18yd\sqrt{z-x}$	$12z\sqrt{da+p}$	$-9m\sqrt{da-y}$

*Exam. 5.* The rational Quantities 13 *y d* and 5 *y d* being added make 18 *y d*, to which joining the *Surd* Quantity  $\sqrt{z-x}$  we have  $18yd\sqrt{z-x}$ , the Sum required.

*Exam. 6.* The rational Quantities  $15z$  and  $-3z$  being added, their Sum by *Art. 3.* is  $12z$ , to which joining the Surd  $\sqrt{da+p}$  we have  $12z\sqrt{da+p}$ , the Sum required.

*Exam. 7.* The rational Quantities  $-7m$  and  $-2m$  being added make  $-9m$ , to which joining  $\sqrt{da-y}$ , we have  $-9m\sqrt{da-y}$ , the Sum required.

To $-2y\sqrt{ma+m}$	$-15m\sqrt{da-zp}$	$16dp\sqrt{14+p}$
Add $-3y\sqrt{ma+m}$	$7m\sqrt{da-zp}$	$-12dp\sqrt{14+p}$
Sum $-5y\sqrt{ma+m}$	$-8m\sqrt{da-zp}$	$4dp\sqrt{14+p}$
To $4y\sqrt{zd-za}$	$5y\sqrt{mp+x}$	$7z\sqrt{ma-d}$
Add $3y\sqrt{zd-za}$	$-4y\sqrt{mp+x}$	$-8z\sqrt{ma-d}$
Sum $7y\sqrt{zd-za}$	$y\sqrt{mp+x}$	$-z\sqrt{ma-d}$

36. *Case 2.* When the Letters under the Radical Sign are different, then place them down one after the other with the same Signs they have in the Question, in the Manner as at *Art. 6.* and this will be the Sum required.

<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
To $\sqrt{a}$	$\sqrt{b+m}$	$m\sqrt{da+y}$
Add $\sqrt{b}$	$\sqrt{d+y}$	$m\sqrt{z}$
Sum $\sqrt{a}+\sqrt{b}$	$\sqrt{b+m}:+\sqrt{d+y}$	$m\sqrt{da+y}:+m\sqrt{z}$

*Exam. 1.* The Letters under the radical Signs being different put down  $\sqrt{a}$ , then because  $\sqrt{b}$  has the Sign  $+$ , therefore after  $\sqrt{a}$  put  $+$ , after which put  $\sqrt{b}$ , and  $\sqrt{a}+\sqrt{b}$ , is the Sum required.

*Exam. 2.* The Letters under the radical Signs being different put down  $\sqrt{b+m}$ : after which place two Dots to show that *Surd* goes no farther, then because  $\sqrt{d+y}$  has the Sign  $+$ , therefore after the Quantity  $\sqrt{b+m}$ : put  $+$ , and after that the Surd  $\sqrt{d+y}$ , and we have  $\sqrt{b+m}:+\sqrt{d+y}$ , the Sum required.

*Exam. 3.* The Letters under the radical Signs being different put down  $m\sqrt{da+y}$ : and because the Quantity  $m\sqrt{z}$  has the Sign  $+$ , therefore after  $m\sqrt{da+y}$ : put the Sign  $+$ , after which put the Quantity  $m\sqrt{z}$ , and we have  $m\sqrt{da+y}:+m\sqrt{z}$ , the Sum required.

*Exam.*

Exam. 4.

$$\begin{array}{r} \text{To } y\sqrt{da} \\ \text{Add } -z\sqrt{m} \\ \hline \text{Sum } y\sqrt{da} - z\sqrt{m} \end{array}$$

Exam. 5.

$$\begin{array}{r} -5\sqrt{da-y} \\ -2m\sqrt{zm} \\ \hline -5\sqrt{da-y} - 2m\sqrt{zm} \end{array}$$

Exam. 6.

$$\begin{array}{r} -2m\sqrt{bz+n} \\ 3y\sqrt{dz-b} \\ \hline -2m\sqrt{bz+n} + 3y\sqrt{dz-b} \end{array}$$

Exam. 4. The Letters under the radical Signs being different put down  $y\sqrt{da}$ , and because  $-z\sqrt{m}$  has the Sign  $-$ , therefore after  $y\sqrt{da}$  put the Sign  $-$ , and after that the Quantity  $z\sqrt{m}$ , and  $y\sqrt{da} - z\sqrt{m}$  is the Sum required.

Exam. 5. The Letters under the radical Signs being different put down  $-5\sqrt{da-y}$ : and because  $-2m\sqrt{zm}$  has the Sign  $-$ , therefore after  $-5\sqrt{da-y}$ : put the Sign  $-$ , and after that the Quantity  $2m\sqrt{zm}$ , and  $-5\sqrt{da-y} - 2m\sqrt{zm}$  is the Sum required.

Exam. 6. Because the Letters under the radical Signs are different I put down  $-2m\sqrt{bz+n}$ , but  $3y\sqrt{dz-b}$  having the Sign  $+$ , therefore after  $-2m\sqrt{bz+n}$ : put the Sign  $+$ , and after that the Quantity  $3y\sqrt{dz-b}$ : and  $-2m\sqrt{bz+n} + 3y\sqrt{dz-b}$  is the Sum required.

$$\begin{array}{r} \text{To } -5\sqrt{da} \\ \text{Add } 7\sqrt{m} \\ \hline \text{Sum } -5\sqrt{da} + 7\sqrt{m} \end{array} \quad \begin{array}{r} m\sqrt{bma} \\ 3\sqrt{yp+q} \\ \hline m\sqrt{bma} + 3\sqrt{yp+q} \end{array}$$

$$\begin{array}{r} \text{To } -3y\sqrt{p+r} \\ \text{Add } m\sqrt{d} \\ \hline \text{Sum } -3y\sqrt{p+r} + m\sqrt{d} \end{array} \quad \begin{array}{r} 14m\sqrt{da+pz} \\ 7\sqrt{z+p} \\ \hline 14m\sqrt{da+pz} + 7\sqrt{z+p} \end{array}$$

$$\begin{array}{r} \text{To } -5y\sqrt{dp-z} \\ \text{Add } 7y\sqrt{zm+a} \\ \hline \text{Sum } -5y\sqrt{dp-z} + 7y\sqrt{zm+a} \end{array}$$

*Subtraction of Surd Quantities, in which there are two Cases.*

37. *Case 1.* When the Letters under the Radical Signs are alike, subtract the rational Quantities from the rational Quantities by Art. 7. and to the Difference join the Surd Quantities, which will be the Remainder required.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
From	$5\sqrt{da}$	$5m\sqrt{mz}$	$14y\sqrt{d+z}$	$21pm\sqrt{db-r}$
Subtract	$3\sqrt{da}$	$2m\sqrt{mz}$	$3y\sqrt{d+z}$	$19pm\sqrt{db-r}$
Remains	$2\sqrt{da}$	$3m\sqrt{mz}$	$11y\sqrt{d+z}$	$2pm\sqrt{db-r}$

*Exam. 1.* The rational Quantities are 5 and 3, subtracting 3 from 5 there remains 2, to which joining the Surd  $\sqrt{da}$  we have  $2\sqrt{da}$ , the Remainder required.

*Exam. 2.* The rational Quantities are  $5m$  and  $2m$ , subtracting  $2m$  from  $5m$  there remains  $3m$ , to which joining the Surd  $\sqrt{mz}$  we have  $3m\sqrt{mz}$ , the Remainder required.

*Exam. 3.* The rational Quantities are  $14y$  and  $3y$ , subtracting  $3y$  from  $14y$  there remains  $11y$ , to which joining the Surd  $\sqrt{d+z}$  we have  $11y\sqrt{d+z}$ , the Remainder required.

*Exam. 4.* The rational Quantities are  $21pm$ , and  $19pm$ , subtracting  $19pm$  from  $21pm$ , there remains  $2pm$ , to which joining the Surd  $\sqrt{db-r}$  we have  $2pm\sqrt{db-r}$ , the Remainder required.

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>
From	$17d\sqrt{ba}$	$-5y\sqrt{d+a}$	$-5m\sqrt{d+ab}$
Subtract	$-4d\sqrt{ba}$	$3y\sqrt{d+a}$	$-6m\sqrt{d+ab}$
Remains	$21d\sqrt{ba}$	$-8y\sqrt{d+a}$	$m\sqrt{d+ab}$

*Exam. 5.* The rational Quantities are  $17d$  and  $-4d$ : Now to subtract  $-4d$  from  $17d$ , by the Rule for Subtraction at Art. 7. change the Sign of  $-4d$ , or suppose it to be changed, then  $-4d$  becomes  $+4d$  or  $4d$ ; then by Art. 7. if we add  $17d$  to  $4d$  it is  $21d$ , which is the Remainder that arises by subtracting  $-4d$  from  $17d$ ; now to this  $21d$  join the Surd  $\sqrt{ba}$ , and  $21d\sqrt{ba}$  is the Remainder required.

*Exam. 6.* To subtract the rational Quantity  $3y$  from  $-5y$ , we must by Art. 7. change or suppose the Sign of  $3y$  to

to be changed, which will make it  $-3y$ : then by the same Art.  $-3y$  added to  $-5y$ , it is  $-8y$ , which is the Remainder that arises from subtracting the rational Quantities, therefore to this  $-8y$  join the Surd Quantity  $\sqrt{d+a}$ , and  $-8y\sqrt{d+a}$  is the Remainder required.

*Exam. 7.* Here the rational Quantities are  $-5m$  and  $-6m$ , and by the Rule for Subtraction Art. 7. if we suppose the Sign of  $-6m$  to be changed, it becomes  $+6m$  or  $6m$ , and then adding  $-5m$  to  $6m$  it is  $m$ , the Remainder arising from subtracting the rational Quantities; and if to this  $m$  we join the Surd  $\sqrt{d+ab}$  we have  $m\sqrt{d+ab}$ , the Remainder required.

	<i>Exam. 8.</i>	<i>Exam. 9.</i>	<i>Exam. 10.</i>
From	$21m\sqrt{d+a}$	$-9d\sqrt{mn+p}$	$12y\sqrt{d-an}$
Subtract	$9m\sqrt{d+a}$	$-2d\sqrt{mn+p}$	$-3y\sqrt{d-an}$
Remains	$12m\sqrt{d+a}$	$-7d\sqrt{mn+p}$	$15y\sqrt{d-an}$

	<i>Exam. 11.</i>	<i>Exam. 12.</i>	<i>Exam. 13.</i>
From	$-4a\sqrt{m-p}$	$14p\sqrt{d-y}$	$-5a\sqrt{x+y}$
Subtract	$2a\sqrt{m-p}$	$-3p\sqrt{d-y}$	$3a\sqrt{x+y}$
Remains	$-6a\sqrt{m-p}$	$17p\sqrt{d-y}$	$-8a\sqrt{x+y}$

	<i>Exam. 14.</i>	<i>Exam. 15.</i>	<i>Exam. 16.</i>
From	$7a\sqrt{ap}$	$-21\sqrt{ap-ax}$	$-14\sqrt{da-z}$
Subtract	$2a\sqrt{ap}$	$-9\sqrt{ap-ax}$	$7\sqrt{da-z}$
Remains	$5a\sqrt{ap}$	$-12\sqrt{ap-ax}$	$-21\sqrt{da-z}$

The Truth of these Operations are proved as in Subtraction of common Numbers. Thus at Example 1. the Remainder is  $2\sqrt{da}$ , and the Quantity subtracted was  $3\sqrt{da}$ , now if we add these together by Art. 35. the Sum is  $5\sqrt{da}$ , which being the same Quantity from which  $3\sqrt{da}$  was subtracted, it proves the Work to be true.

Again, at Example 6. the Remainder is  $-8y\sqrt{d+a}$ : the Quantity subtracted was  $3y\sqrt{d+a}$ : Now by Art. 35. if to  $-8y\sqrt{d+a}$  we add  $3y\sqrt{d+a}$ , the Sum is  $-5y\sqrt{d+a}$ , which being the Quantity from which  $3y\sqrt{d+a}$  was subtracted, it proves the Work to be true.

38. *Case 3.* When the Letters under the radical Signs are different, set them down one after the other, as at Art. 36. but in setting them down take Care to change the Signs of those Quantities that are to be subtracted, by Art. 7. and this will be the Remainder required.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
From	$2\sqrt{da}$	$2m\sqrt{dp}$	$5y\sqrt{a}$
Subtract	$3\sqrt{m}$	$y\sqrt{z}$	$-3\sqrt{b}$
Remains	$2\sqrt{da} - 3\sqrt{m}$	$2m\sqrt{dp} - y\sqrt{z}$	$5y\sqrt{a} + 3\sqrt{b}$

*Exam. 1.* The Letters under the radical Signs being different place down  $2\sqrt{da}$ , and because  $3\sqrt{m}$  the Quantity to be subtracted has the Sign +, therefore after  $2\sqrt{da}$  place the Sign —, and after that the Quantity  $3\sqrt{m}$ , and  $2\sqrt{da} - 3\sqrt{m}$  is the Remainder required.

*Exam. 2.* Because the Letters under the radical Signs are different put down  $2m\sqrt{dp}$ , and because  $y\sqrt{z}$  the Quantity to be subtracted has the Sign +, therefore after  $2m\sqrt{dp}$  put the Sign —, and after that  $y\sqrt{z}$ , and  $2m\sqrt{dp} - y\sqrt{z}$  is the Remainder required.

*Exam. 3.* Because the Letters under the radical Signs are different put down  $5y\sqrt{a}$ , but as  $-3\sqrt{b}$  the Quantity to be subtracted has the Sign —, therefore after  $5y\sqrt{a}$  put the Sign +, and after that  $3\sqrt{b}$ , and  $5y\sqrt{a} + 3\sqrt{b}$  is the Remainder required.

	<i>Exam. 4.</i>	<i>Exam. 5.</i>
From	$m\sqrt{da+p}$	$-5y\sqrt{a}$
Subtract	$2\sqrt{a}$	$-d\sqrt{b}$
Remains	$m\sqrt{da+p} - 2\sqrt{a}$	$-5y\sqrt{a} + d\sqrt{b}$

*Exam. 6.*

From	$5m\sqrt{a}$
Subtract	$-z\sqrt{p+q}$
Remains	$5m\sqrt{a} + z\sqrt{p+q}$

*Exam. 4.* Because the Letters under the radical Signs are different put down  $m\sqrt{da+p}$ , but as  $2\sqrt{a}$  the Quantity to be subtracted has the Sign +, therefore after  $m\sqrt{da+p}$  put the Sign —, and after that  $2\sqrt{a}$ , and  $m\sqrt{da+p} - 2\sqrt{a}$  is the Remainder required.

*Exam.*

*Exam. 5.* Because the Letters under the radical Signs are different put down  $5y\sqrt{a}$ , but as  $-d\sqrt{b}$  the Quantity to be subtracted has the Sign  $-$ , therefore after  $5y\sqrt{a}$  put the Sign  $+$ , and after that  $d\sqrt{b}$ , and  $-5y\sqrt{a} + d\sqrt{b}$  is the Remainder required.

*Exam. 6.* Because the Letters under the radical Signs are different put down  $5m\sqrt{a}$ , but as  $-z\sqrt{p+q}$  has the Sign  $-$  before it, therefore after  $5m\sqrt{a}$  put the Sign  $-$ , and after that  $z\sqrt{p+q}$ , and  $5m\sqrt{a} + z\sqrt{p+q}$  is the Remainder required.

$$\begin{array}{r} \text{From } 5\sqrt{a+p} \\ \text{Subtract } m\sqrt{y} \\ \hline \text{Remains } 5\sqrt{a+p} : -m\sqrt{y} \end{array} \qquad \begin{array}{r} m\sqrt{p} \\ -y\sqrt{da-p} \\ \hline m\sqrt{p+y}\sqrt{da-p} \end{array}$$

$$\begin{array}{r} \text{From } 3u\sqrt{d+p} \\ \text{Subtract } 2n\sqrt{z-y} \\ \hline \text{Remains } 3u\sqrt{d+p} : -2n\sqrt{z-y} \end{array} \qquad \begin{array}{r} -5n\sqrt{da} \\ -3\sqrt{m} \\ \hline -5n\sqrt{da} + 3\sqrt{m} \end{array}$$

$$\begin{array}{r} \text{From } -5\sqrt{p+z} \\ \text{Subtract } 3n\sqrt{m} \\ \hline \text{Remains } -5\sqrt{p+z} : -3n\sqrt{m} \end{array} \qquad \begin{array}{r} 14\sqrt{da} \\ 7\sqrt{p+y} \\ \hline 14\sqrt{da} : -7\sqrt{p+y} \end{array}$$

The Truth of these Operations are proved in the same Manner as in the last Article, by adding the Remainder to the Quantity that was subtracted; and if their Sum makes the Quantity from which the other was taken, the Work is true, if not, there is a Mistake.

Thus at *Example 1.* the Remainder is  $2\sqrt{da} - 3\sqrt{m}$   
To which if we add the Quantity }  
subtracted  $3\sqrt{m}$

The Sum is  $2\sqrt{da}$ , the same in the given Example. For in this Addition, adding  $+3\sqrt{m}$  to  $-3\sqrt{m}$ , the Co-efficients and Quantities being the same and the Signs contrary, they destroy one another, or the Sum is *nothing*, by Art. 5.

Again at *Example 5.* the Remainder is  $-5y\sqrt{a} + d\sqrt{b}$   
To which if we add the Quantity }  
subtracted  $-d\sqrt{b}$

The Sum is  $-5y\sqrt{a}$ , the same as in the given Example. For here  $-d\sqrt{b}$  being added to  $d\sqrt{b}$  or  $+d\sqrt{b}$ , they destroy one another

another as in the last Instance. In like Manner the Reader may prove any of the other Examples.

*Multiplication of Surd Quantities, in which there are two Cases.*

39. *Case 1.* When there are no rational Quantities but Unity joined to the Surd Quantities, then multiply the Surd Quantities, as in Multiplication of rational Quantities, but to their Product prefix the radical Sign.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Multiply	$\sqrt{a}$	$\sqrt{mn}$	$\sqrt{py}$	$\sqrt{zx}$
By	$\sqrt{m}$	$\sqrt{d}$	$\sqrt{z}$	$\sqrt{a}$
Product	$\sqrt{am}$	$\sqrt{mnd}$	$\sqrt{pyz}$	$\sqrt{zxa}$

*Exam. 1.* Multiplying  $a$  by  $m$ , the Product is  $am$ , to which prefixing the Sign  $\sqrt{\phantom{x}}$ , we have  $\sqrt{am}$  the Product required.

*Exam. 2.* Multiplying  $mn$  by  $d$ , the Product is  $mnd$ , to which prefixing the Sign  $\sqrt{\phantom{x}}$ , we have  $\sqrt{mnd}$ , the Product required.

*Exam. 3.* Multiplying  $py$  by  $z$ , the Product is  $pyz$ , to which prefixing the Sign  $\sqrt{\phantom{x}}$ , we have  $\sqrt{pyz}$ , the Product required.

*Exam. 4.* Multiplying  $zx$  by  $a$ , the Product is  $zxa$ , to which prefixing the Sign  $\sqrt{\phantom{x}}$ , we have  $\sqrt{zxa}$ , the Product required.

	<i>Exam. 5.</i>	<i>Exam. 6.</i>
Multiply	$\sqrt{pa}$	$\sqrt{zy}$
By	$\sqrt{z}$	$\sqrt{x}$
Product	$\sqrt{paz}$	$\sqrt{zyx}$

	<i>Exam. 7.</i>	<i>Exam. 8.</i>
Multiply	$\sqrt{a+b}$	$\sqrt{mn - z}$
By	$\sqrt{y}$	$\sqrt{a}$
Product	$\sqrt{ay + yb}$	$\sqrt{mna - az}$

*Exam. 7.* Multiplying  $a + b$  by  $y$ , the Product is  $ay + yb$ , by Art. 10. to which prefixing the Sign  $\sqrt{\phantom{x}}$ , and drawing it over all the Quantities, we have  $\sqrt{ay + yb}$ , the Product required.

*Exam.*



*Exam. 8.* Multiplying  $mn - z$  by  $a$  the Product is  $mna - az$ , by Art. 10 and 16. to which prefixing the radical Sign as in the last Example, we have  $\sqrt{mna - az}$  the Product required.

<i>Exam. 9.</i>	<i>Exam. 10.</i>	<i>Exam. 11.</i>
Multiply $\sqrt{ap + z}$	$\sqrt{az - ap}$	$\sqrt{d - y}$
By $\sqrt{y}$	$\sqrt{m}$	$\sqrt{p}$
Product $\sqrt{apy + yz}$	$\sqrt{azm - apm}$	$\sqrt{dp - py}$

*Exam. 9.* Multiplying  $ap + z$  by  $y$ , the Product is  $apy + yz$ , to which prefixing the radical Sign, we have  $\sqrt{apy + yz}$  the Product required.

*Exam. 10.* Multiplying  $az - ap$  by  $m$ , the Product is  $azm - apm$ , to which prefixing the radical Sign, we have  $\sqrt{azm - apm}$  the Product required.

*Exam. 11.* Multiplying  $d - y$  by  $p$ , the Product is  $dp - py$ , to which prefixing the radical Sign, we have  $\sqrt{dp - py}$  the Product required.

Multiply $\sqrt{ab}$	$\sqrt{zy}$	$\sqrt{ap - z}$
By $\sqrt{a - p}$	$\sqrt{d + y}$	$\sqrt{d}$
Product $\sqrt{aab - abp}$	$\sqrt{zyd + zyy}$	$\sqrt{apd - dz}$

Multiply $\sqrt{ap + z}$	$\sqrt{ay}$	$\sqrt{m}$
By $\sqrt{m}$	$\sqrt{d - z}$	$\sqrt{a - py}$
Product $\sqrt{apm + zm}$	$\sqrt{ayd - ayz}$	$\sqrt{ma - mpy}$

40. *Case 2.* When there are rational Quantities joined to the Surds, then multiply the rational Quantities together as in Multiplication of rational Quantities, after which multiply the Surd Quantities together by the last Article, and joining these two Products, this will be the Product required.

If there are no rational Quantities prefixt, then Unity, or 1, is always supposed to be the rational Quantity.

<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Multiply $a\sqrt{m}$	$ap\sqrt{z}$	$3\sqrt{mn}$	$\sqrt{mp}$
By $d\sqrt{y}$	$2\sqrt{a}$	$a\sqrt{p}$	$y\sqrt{d}$
Product $ad\sqrt{my}$	$2ap\sqrt{za}$	$3a\sqrt{mnp}$	$y\sqrt{mpd}$

K

*Exam.*

*Exam. 1.* Multiplying the rational Quantities  $a$  and  $d$ , the Product is  $ad$ , and multiplying the Surds  $\sqrt{m}$  by  $\sqrt{y}$ , the Product is  $\sqrt{my}$  by Art. 39. joining this to the rational Quantity  $ad$ , we have  $ad\sqrt{my}$ , the Product required.

*Exam. 2.* Multiplying the rational Quantities  $ap$  by 2, the Product is  $2ap$ , and multiplying the Surds  $\sqrt{z}$  by  $\sqrt{a}$ , the Product is  $\sqrt{za}$  by Art. 39. joining these, and  $2ap\sqrt{za}$  is the Product required.

*Exam. 3.* Multiplying the rational Quantities 3. and  $a$ , the Product is  $3a$ , and multiplying the Surds  $\sqrt{mn}$  by  $\sqrt{p}$ , the Product is  $\sqrt{mnp}$ , by Art. 39. joining these we have  $3a\sqrt{mnp}$ , the Product required.

*Exam. 4.* Multiplying the rational Quantities  $y$  and 1, (for 1 is the rational Quantity of  $\sqrt{mp}$ , there being no rational Quantity prefixt) the Product is  $y$ , and multiplying the Surds  $\sqrt{mp}$  by  $\sqrt{d}$ , the Product is  $\sqrt{mpd}$  by Art. 39. and joining these we have  $y\sqrt{mpd}$ , the Product required.

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
Multiply	$am\sqrt{p}$	$y\sqrt{pq}$	$m\sqrt{2p}$	$2a\sqrt{3z}$
By	$z\sqrt{d}$	$d\sqrt{z}$	$4y\sqrt{y}$	$3d\sqrt{4y}$
Product	$amz\sqrt{pd}$	$yd\sqrt{pqz}$	$4my\sqrt{2py}$	$6ad\sqrt{12zy}$

*Exam. 5.* The Product of the rational Quantities is  $amz$ , and the Product of the Surds is  $\sqrt{pd}$ , these being joined we have  $amz\sqrt{pd}$ , the Product required.

*Exam. 6.* The Product of the rational Quantities is  $yd$ , and the Product of the Surds is  $\sqrt{pqz}$ , these being joined we have  $yd\sqrt{pqz}$ , the Product required.

*Exam. 7.* The Product of the rational Quantities is  $4my$ , and the Product of the Surds is  $\sqrt{2py}$ , these being joined we have  $4my\sqrt{2py}$ , the Product required.

*Exam. 8.* The Product of the rational Quantities is  $6ad$ , and the Product of the Surds is  $\sqrt{12zy}$ , these being joined we have  $6ad\sqrt{12zy}$ , the Product required.

	<i>Exam. 9.</i>	<i>Exam. 10.</i>	<i>Exam. 11.</i>	<i>Exam. 12.</i>
Multiply	$y\sqrt{p}$	$\sqrt{mn}$	$2\sqrt{dx}$	$3\sqrt{2z}$
By	$a\sqrt{z}$	$a\sqrt{y}$	$\sqrt{z}$	$5\sqrt{7y}$
Product	$ya\sqrt{pz}$	$a\sqrt{mny}$	$2\sqrt{dxz}$	$15\sqrt{14yz}$

*Exam.*

*Exam. 13.*

$$\begin{array}{l} \text{Multiply } m\sqrt{a+y} \\ \text{By } a\sqrt{p} \\ \hline \text{Product } ma\sqrt{ap+py} \end{array}$$

*Exam. 14.*

$$\begin{array}{l} d\sqrt{m-pz} \\ y\sqrt{d} \\ \hline dy\sqrt{md-pzd} \end{array}$$

*Exam. 15.*

$$\begin{array}{l} a\sqrt{ap+z} \\ x\sqrt{y} \\ \hline ax\sqrt{apy+yz} \end{array}$$

*Exam. 13.* Multiplying the rational Quantities  $m$  and  $a$ , the Product is  $ma$ , and multiplying  $a+y$  by  $p$ , the Product is  $ap+py$ , but prefixing to this the Sign  $\sqrt{\phantom{x}}$ , because they are Surds, we have  $\sqrt{ap+py}$  for the Product of the Surds, which joining to  $ma$  the Product of the rational Quantities, we have  $ma\sqrt{ap+py}$ , the Product required.

*Exam. 14.* Multiplying the rational Quantities  $d$  and  $y$ , the Product is  $dy$ , and multiplying the Surds  $\sqrt{m-pz}$  by  $\sqrt{d}$ , the Product is  $\sqrt{md-pzd}$ , which being joined to  $dy$ , the Product of the rational Quantities, we have  $dy\sqrt{md-pzd}$ , the Product required.

*Exam. 15.* The Product of the rational Quantities is  $ax$ , and the Product of the Surds is  $\sqrt{apy+yz}$ , these being joined we have  $ax\sqrt{apy+yz}$ , the Product required.

*Exam. 16.*

$$\begin{array}{l} \text{Multiply } am\sqrt{py+d} \\ \text{By } y\sqrt{z} \\ \hline \text{Product } amy\sqrt{pyz+zd} \end{array}$$

*Exam. 17.*

$$\begin{array}{l} 2\sqrt{am-y} \\ a\sqrt{p} \\ \hline 2a\sqrt{amp-py} \end{array}$$

*Exam. 18.*

$$\begin{array}{l} \text{Multiply } m\sqrt{pd} \\ \text{By } a\sqrt{d-a} \\ \hline \text{Product } ma\sqrt{pdd-pda} \end{array}$$

*Exam. 16.* The rational Quantities  $am$  and  $y$  being multiplied, the Product is  $amy$ , and  $py+d$  being multiplied by  $z$ , the Product is  $pyz+zd$ ; but before it prefix the radical Sign, because these Quantities are Surds, then it is  $\sqrt{pyz+zd}$ , joining this to the Product  $amy$ , we have  $amy\sqrt{pyz+zd}$ , the Product required.

*Exam. 17.* The Product of the rational Quantities 2 and  $a$ , is  $2a$ , and the Product of the Surds is  $\sqrt{amp - py}$ : joining these we have  $2a\sqrt{amp - py}$ , the Product required.

*Exam. 18.* The Product of the rational Quantities  $m$  and  $a$ , is  $ma$ , and  $pd$  multiplied into  $d - a$ , is  $pd d - pda$ , to which prefix the radical Sign, because these are Surds, and this becomes  $\sqrt{pd d - pda}$ , now joining it to  $ma$ , we have  $ma\sqrt{pd d - pda}$ , the Product required.

$$\begin{array}{r} \text{Multiply } a\sqrt{p-y} \\ \text{By } 2b\sqrt{m} \\ \hline \text{Product } 2a\sqrt{bpm-my} \end{array} \quad \begin{array}{r} 3\sqrt{m-n} \\ 2\sqrt{5a} \\ \hline 6\sqrt{5ma-5na} \end{array} \quad \begin{array}{r} \sqrt{n+b} \\ a\sqrt{z} \\ \hline a\sqrt{zn+zb} \end{array}$$

$$\begin{array}{r} \text{Multiply } 2a\sqrt{3y+z} \\ \text{By } b\sqrt{d} \\ \hline \text{Product } 2ab\sqrt{3yd+dz} \end{array} \quad \begin{array}{r} y\sqrt{p-z} \\ 3m\sqrt{a} \\ \hline 3ym\sqrt{pa-az} \end{array}$$

$$\begin{array}{r} \text{Multiply } 5\sqrt{y-x} \\ \text{By } 3a\sqrt{2b} \\ \hline \text{Product } 15a\sqrt{2by-2bx} \end{array}$$

*Division of Surd Quantities, in which there are two Cases.*

41. *Case 1.* When there are no rational Quantities joined with the Surd Quantities, reject all those Quantities in the Dividend and Divisor that are alike, as at Art. 20. and set down the Remainder, to which prefix the radical Sign, and this will be the Quotient sought.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Divide	$\sqrt{mn}$	$\sqrt{ma}$	$\sqrt{abd}$	$\sqrt{abd}$
By	$\sqrt{m}$	$\sqrt{a}$	$\sqrt{ab}$	$\sqrt{a}$
Quotient	$\sqrt{n}$	$\sqrt{m}$	$\sqrt{d}$	$\sqrt{bd}$

*Exam. 1.* Because  $m$  is in both Dividend and Divisor, reject it, and place down  $n$  the remaining Part of the Dividend, to which prefixing the radical Sign, and  $\sqrt{n}$  is the Quotient required.

*Exam.*

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*Exam. 2.* Because  $a$  is in both Dividend and Divisor, reject it, and place down  $m$  the remaining Part of the Dividend, to which prefixing the radical Sign, we have  $\sqrt{m}$ , the Quotient required.

*Exam. 3.* Because  $ab$  is in both Dividend and Divisor, reject it, and place down  $d$  the remaining Part of the Dividend, to which prefixing the radical Sign, we have  $\sqrt{d}$ , the Quotient required.

*Exam. 4.* Because  $a$  is in both Dividend and Divisor, reject it, and place down  $bd$  the remaining Part of the Dividend, to which prefixing the radical Sign, we have  $\sqrt{bd}$ , the Quotient required.

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
Divide	$\sqrt{m d y}$	$\sqrt{b z d}$	$\sqrt{b z d}$	$\sqrt{y p a}$
By	$\sqrt{y}$	$\sqrt{b d}$	$\sqrt{z d}$	$\sqrt{y p}$
Quotient	$\sqrt{m d}$	$\sqrt{z}$	$\sqrt{b}$	$\sqrt{a}$

*Exam. 5.* Because  $y$  is in both Dividend and Divisor, reject it, and place down  $md$  with the Sign  $\sqrt{\phantom{x}}$  before it, and  $\sqrt{md}$  is the Quotient required.

*Exam. 6.* Because  $bd$  is in both Dividend and Divisor, reject it, and place down  $z$  with the Sign  $\sqrt{\phantom{x}}$  before it, and  $\sqrt{z}$  is the Quotient required.

*Exam. 7.* Because  $zd$  is in both Dividend and Divisor, reject it, and place down  $b$  with the Sign  $\sqrt{\phantom{x}}$  before it, and we have  $\sqrt{b}$ , the Quotient required.

*Exam. 8.* Because  $yp$  is in both Dividend and Divisor, reject it, and place down  $a$  with the Sign  $\sqrt{\phantom{x}}$  before it, and  $\sqrt{a}$  is the Quotient required.

	<i>Exam. 9.</i>	<i>Exam. 10.</i>	<i>Exam. 11.</i>	<i>Exam. 12.</i>
Divide	$\sqrt{m a x}$	$\sqrt{n d y}$	$\sqrt{a y z}$	$\sqrt{a b d}$
By	$\sqrt{x}$	$\sqrt{n y}$	$\sqrt{a}$	$\sqrt{a d}$
Quotient	$\sqrt{m a}$	$\sqrt{d}$	$\sqrt{y z}$	$\sqrt{b}$

	<i>Exam. 13.</i>	<i>Exam. 14.</i>	<i>Exam. 15.</i>
Divide	$\sqrt{a m + a p}$	$\sqrt{p y - p n}$	$\sqrt{b d - b m}$
By	$\sqrt{a}$	$\sqrt{p}$	$\sqrt{b}$
Quotient	$\sqrt{m + p}$	$\sqrt{y - n}$	$\sqrt{d - m}$

*Exam.*

*Exam. 13.* If we divide  $am + ap$  by  $a$ , the Quotient is  $m + p$  by Art. 22. but because they are Surds, prefix the Sign  $\sqrt{\phantom{x}}$  to  $m + p$ , and  $\sqrt{m + p}$  is the Quotient required.

*Exam. 14.* Dividing  $py - pn$  by  $p$ , the Quotient is  $y - n$ , by Art. 22 and 24. to which prefixing the Sign  $\sqrt{\phantom{x}}$ , we have  $\sqrt{y - n}$ , the Quotient required.

*Exam. 15.* Dividing  $bd - bm$  by  $b$ , the Quotient is  $d - m$ , by Art. 22 and 24. to which prefixing the Sign  $\sqrt{\phantom{x}}$ , we have  $\sqrt{d - m}$ , the Quotient required.

Divide $\sqrt{bn + ba}$ By $\sqrt{b}$ <hr style="width: 100%;"/> Quotient $\sqrt{n + a}$	Divide $\sqrt{mx - md}$ By $\sqrt{m}$ <hr style="width: 100%;"/> Quotient $\sqrt{x - d}$	Divide $\sqrt{nz - np}$ By $\sqrt{n}$ <hr style="width: 100%;"/> Quotient $\sqrt{z - p}$
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Divide $\sqrt{zx - zy}$ By $\sqrt{z}$ <hr style="width: 100%;"/> Quotient $\sqrt{x - y}$	Divide $\sqrt{ad + ay}$ By $\sqrt{a}$ <hr style="width: 100%;"/> Quotient $\sqrt{d + y}$	Divide $\sqrt{bd - bm}$ By $\sqrt{b}$ <hr style="width: 100%;"/> Quotient $\sqrt{d - m}$
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The Truth of these Operations are proved by multiplying the Quotient by the Divisor, for if that produces the Dividend, the Work is true, otherwise it is erroneous. Thus in Example 2, Page 68. the Divisor is  $\sqrt{a}$ , and the Quotient is  $\sqrt{m}$ , which being multiplied by Art. 39. the Product is  $\sqrt{ma}$ , the given Dividend.

And at Example 6, Page 69. the Divisor is  $\sqrt{bd}$ , and the Quotient is  $\sqrt{z}$ , which being multiplied by Art. 39. the Product is  $\sqrt{bzd}$ , the given Dividend.

And at Example 13, the Divisor is  $\sqrt{a}$ , and the Quotient is  $\sqrt{m + p}$ , which being multiplied by Art. 39. the Product is  $\sqrt{am + ap}$ , the given Dividend; in the same Manner may any of the other Examples be proved.

42. *Case 2.* When there are rational Quantities joined with the Surds, divide the rational Quantities by the rational Quantities, by the Rules in Division of rational Quantities; and to their Quotient, join the Quotient of the Surds found by the last Article, which will be the Quotient required.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Divide	$ay\sqrt{mn}$	$bm + yz$	$yd\sqrt{az}$	$ma\sqrt{ayn}$
By	$a\sqrt{m}$	$m\sqrt{z}$	$y\sqrt{a}$	$a\sqrt{ay}$
Quotient	$y\sqrt{n}$	$b\sqrt{y}$	$d\sqrt{z}$	$m\sqrt{n}$
	$\frac{1}{2}$			<i>Exam.</i>

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*Exam. 1.* Dividing the rational Quantities  $ay$  by  $a$ , the Quotient is  $y$  by Art. 20. and dividing  $\sqrt{mn}$  by  $\sqrt{m}$ , the Quotient is  $\sqrt{n}$  by Art. 41. now joining  $y$  to  $\sqrt{n}$ , we have  $y\sqrt{n}$ , the Quotient required.

*Exam. 2.* Dividing the rational Quantities  $bm$  by  $m$ , the Quotient is  $b$  by Art. 20. and dividing  $\sqrt{yz}$  by  $\sqrt{z}$ , the Quotient is  $\sqrt{y}$  by Art. 41. now joining  $b$  and  $\sqrt{y}$ , we have  $b\sqrt{y}$ , the Quotient required.

*Exam. 3.* Dividing the rational Quantities  $yd$  by  $y$ , the Quotient is  $d$  by Art. 20. and dividing  $\sqrt{az}$  by  $\sqrt{a}$ , the Quotient is  $\sqrt{z}$  by Art. 41. now joining  $d$  and  $\sqrt{z}$ , we have  $d\sqrt{z}$ , the Quotient required.

*Exam. 4.* Dividing the rational Quantities  $ma$  by  $a$ , the Quotient is  $m$  by Art. 20. and dividing  $\sqrt{ayn}$  by  $\sqrt{ay}$ , the Quotient is  $\sqrt{n}$  by Art. 41. now joining  $m$  and  $\sqrt{n}$ , we have  $m\sqrt{n}$ , the Quotient required.

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
Divide	$ayn\sqrt{mn}$	$mn\sqrt{xay}$	$xa\sqrt{nd}$	$dz\sqrt{anp}$
By	$ay\sqrt{m}$	$n\sqrt{xy}$	$a\sqrt{n}$	$z\sqrt{an}$
Quotient	$n\sqrt{n}$	$m\sqrt{a}$	$x\sqrt{d}$	$d\sqrt{p}$

*Exam. 5.* Dividing the rational Quantities  $ayn$  by  $ay$ , the Quotient is  $n$  by Art. 20. and dividing  $\sqrt{mn}$  by  $\sqrt{m}$ , the Quotient is  $\sqrt{n}$  by Art. 41. now joining  $n$  and  $\sqrt{n}$ , we have  $n\sqrt{n}$ , the Quotient required.

*Exam. 6.* Dividing the rational Quantities  $mn$  by  $n$ , the Quotient is  $m$  by Art. 20. and dividing  $\sqrt{xay}$  by  $\sqrt{xy}$ , the Quotient is  $\sqrt{a}$  by Art. 41. now joining  $m$  and  $\sqrt{a}$ , we have  $m\sqrt{a}$ , the Quotient required.

*Exam. 7.* Dividing the rational Quantities  $xa$  by  $a$ , the Quotient is  $x$ , and dividing  $\sqrt{nd}$  by  $\sqrt{n}$ , the Quotient is  $\sqrt{d}$  by Art. 41. now joining  $x$  and  $\sqrt{d}$ , we have  $x\sqrt{d}$ , the Quotient required.

*Exam. 8.* Dividing the rational Quantities  $dz$  by  $z$ , the Quotient is  $d$ , and dividing  $\sqrt{anp}$  by  $\sqrt{an}$ , the Quotient is  $\sqrt{p}$  by Art. 41. now joining  $d$  and  $\sqrt{p}$ , we have  $d\sqrt{p}$ , the Quotient required.

	<i>Exam. 9.</i>	<i>Exam. 10.</i>	<i>Exam. 11.</i>	<i>Exam. 12.</i>
Divide	$4mn\sqrt{ab}$	$my\sqrt{az}$	$dn\sqrt{xy}$	$8az\sqrt{rd}$
By	$2m\sqrt{a}$	$y\sqrt{z}$	$n\sqrt{x}$	$4a\sqrt{r}$
Quotient	$2n\sqrt{b}$	$m\sqrt{a}$	$d\sqrt{y}$	$2n\sqrt{d}$

*Exam.*

	<i>Exam. 13.</i>	<i>Exam. 14.</i>	<i>Exam. 15.</i>	<i>Exam. 16.</i>
Divide	$m x \sqrt{pq}$	$4 a n \sqrt{rd}$	$x z \sqrt{m y p}$	$r m \sqrt{dz}$
By	$x \sqrt{p}$	$a \sqrt{d}$	$z \sqrt{m y}$	$r \sqrt{d}$
Quotient	$m \sqrt{q}$	$4 n \sqrt{r}$	$x \sqrt{p}$	$m \sqrt{z}$

	<i>Exam. 17.</i>	<i>Exam. 18.</i>	<i>Exam. 19.</i>
Divide	$m n \sqrt{ap + ax}$	$y p \sqrt{zd + zm}$	$d a y \sqrt{ym + yr}$
By	$m \sqrt{a}$	$p \sqrt{z}$	$d a \sqrt{y}$
Quotient	$n \sqrt{p + x}$	$y \sqrt{d + m}$	$y \sqrt{m + r}$

*Exam. 17.* Dividing the rational Quantities  $m n$  by  $m$ , the Quotient is  $n$  by Art. 20. and dividing  $\sqrt{ap + ax}$  by  $\sqrt{a}$ , we have  $\sqrt{p + x}$  by Art. 41. and joining  $n$  and  $\sqrt{p + x}$ , we have  $n \sqrt{p + x}$ , the Quotient required.

*Exam. 18.* Dividing the rational Quantities  $y p$  by  $p$ , the Quotient is  $y$  by Art. 20. and dividing  $\sqrt{zd + zm}$  by  $\sqrt{z}$ , the Quotient is  $\sqrt{d + m}$  by Art. 41. joining  $y$  and  $\sqrt{d + m}$ , we have  $y \sqrt{d + m}$ , the Quotient required.

*Exam. 19.* Dividing the rational Quantities  $d a y$  by  $d a$ , the Quotient is  $y$  by Art. 20. and dividing  $\sqrt{ym + yr}$  by  $\sqrt{y}$ , the Quotient is  $\sqrt{m + r}$  by Art. 41. joining  $y$  and  $\sqrt{m + r}$ , we have  $y \sqrt{m + r}$ , the Quotient required. The following Examples are done in the same Manner.

Divide	$4 a n \sqrt{dy + dn}$	$a n \sqrt{pz - pb}$	$6 b d y \sqrt{pm + pd}$
By	$2 a \sqrt{d}$	$a \sqrt{p}$	$3 b d \sqrt{p}$
Quotient	$2 n \sqrt{y + n}$	$n \sqrt{z - b}$	$2 y \sqrt{m + d}$

Divide	$p n \sqrt{dx - db}$	$12 b a \sqrt{py - px}$	$a n x \sqrt{pd - pm}$
By	$n \sqrt{d}$	$3 a \sqrt{p}$	$a x \sqrt{p}$
Quotient	$p \sqrt{x - b}$	$4 b \sqrt{y - x}$	$n \sqrt{d - m}$

The Truth of these Operations are proved likewise from multiplying the Quotient by the Divisor, and if that Product makes the Dividend, the Work is true, if not, there is a Mistake. Thus in

*Exam.*



*Exam. 1.* The Quotient is  $y\sqrt{n}$ , and the Divisor is  $a\sqrt{m}$ ; now multiplying  $y\sqrt{n}$  by  $a\sqrt{m}$ , by Art. 40. first multiply the rational Quantities  $y$  and  $a$ , this Product is  $ay$ , and multiplying  $\sqrt{n}$  by  $\sqrt{m}$ , this Product is  $\sqrt{mn}$ , and joining this to  $ay$  we have  $ay\sqrt{mn}$  the Product, which being the same as the given Dividend, proves the Work to be true.

And at *Exam. 5.* the Divisor is  $ay\sqrt{m}$ , the Quotient is  $n\sqrt{n}$ , now multiplying  $ay\sqrt{m}$  by  $n\sqrt{n}$ , according to Art. 40. we first multiply the rational Quantities  $ay$  by  $n$ , and this Product is  $ayn$ ; then multiplying  $\sqrt{m}$  by  $\sqrt{n}$ , this Product is  $\sqrt{mn}$ , and joining this to  $ayn$ , the Product is  $ayn\sqrt{mn}$ , which being the same with the given Dividend, the Work is true.

And at *Exam. 17.* the Divisor is  $m\sqrt{a}$ , and the Quotient is  $n\sqrt{p+x}$ , and multiplying these by Art. 40. we first multiply the rational Quantities  $m$  and  $n$  together, and this Product is  $mn$ , then multiplying  $\sqrt{a}$  by  $\sqrt{p+x}$ , this Product is  $\sqrt{ap+ax}$ , which being joined to  $mn$ , the Product is  $mn\sqrt{ap+ax}$ , the same as the given Dividend; and so may any of the other Examples be proved.

### Involution of Surd Quantities.

43. *Case 1.* When there are no rational Quantities joined with the Surds, it is only setting the Quantities down without their radical Sign, which raises the given Root as high as is the Index of the radical Sign.

*Exam. 1. Exam. 2. Exam. 3. Exam. 4.*

Raise to the Square } or second Power }	$\sqrt{a}$	$\sqrt{mn}$	$\sqrt{na}$	$\sqrt{b}$
The Square	$a$	$mn$	$na$	$b$

This being no more, according to the Rule, but to set down the Quantities without their radical Sign, it is so easy as not to want any farther Explanation.

The Reason on which the Operation is founded, is, that any Quantity or Number being multiplied by itself, will produce the Square of that Quantity or Number, thus  $2 \times 2 = 4$ , whence 4 is the Square of 2, and  $a \times a = aa$ , which is the Square of  $a$ , and so of any other Quantity. Now supposing the Square Root of  $a$  was to be extracted, which by Art. 33. is  $\sqrt{a}$ . But

as  $\sqrt{a}$  is the Root, and  $a$  was the Square from which that Root was extracted, hence  $\sqrt{a}$  multiplied into  $\sqrt{a}$ , must produce  $a$ , by what has been just said: Now  $\sqrt{a}$  multiplied by  $\sqrt{a}$ , is  $\sqrt{aa}$  by Art. 39. and as  $\sqrt{aa}$  signifies the Square Root of  $aa$ , which is  $a$  by Art. 33. it follows, that to involve any Surd that has no rational Quantities joined with it, is only to set down the Quantities without their radical Sign.

To find the Square or second Power of	$\sqrt{ax}$	$\sqrt{nd}$	$\sqrt{pr}$	$\sqrt{z}$
The Square	$ax$	$nd$	$pr$	$z$

And if there are several Quantities connected by the Signs  $+$  or  $-$ , and are all under the radical Sign, they are involved in the same Manner.

Raise to the second Power or Square	$\sqrt{a+b}$	$\sqrt{an-d}$	$\sqrt{p+ny}$
The Square	$a+b$	$an-d$	$p+ny$

Raise to the second Power or Square	$\sqrt{pd-n}$	$\sqrt{dz+zy}$	$\sqrt{pm-nd}$
The Square	$pd-n$	$dz+zy$	$pm-nd$

Raise to the 2d Power or Square	$\sqrt{a+y-d}$	$\sqrt{am-n+db}$	$\sqrt{pz+zx-xd}$
The Square	$a+y-d$	$am-n+db$	$pz+zx-xd$

44. *Case 2.* When there are rational Quantities joined with the Surds, then involve the rational Quantities as high as is the Index of the Surd, and multiply these involved Quantities into the Surd Quantities, after the radical Sign is taken away.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Raise to the Square	$a\sqrt{m}$	$b\sqrt{nz}$	$d\sqrt{y}$	$zz\sqrt{b}$
The Square	$aam$	$bbnz$	$ddy$	$zzzzb$

*Ex. 1.* The rational Quantity  $a$  squared is by Art. 31.  $aa$   
 The Surd Quantity  $\sqrt{m}$  being put down }  
 without the radical Sign is }  $-$   $-$   $m$   
 These being multiplied, the Product is the Square required  $aam$

*Exam.*

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*Ex. 2.* The rational Quantity  $b$  squared is by Art. 31.  $bb$   
 The Surd Quantity  $\sqrt[n]{z}$  without the radical Sign is  $nz$   
 These multiplied, the Product is the Square required  $bbnz$

*Exam. 3.* The rational Quantity  $d$  squared is  $dd$   
 The Surd Quantity  $\sqrt{y}$  without the radical Sign is  $y$   
 These multiplied, the Product is the Square required  $ddy$

*Exam. 4.* The rational Quantity  $zz$  squared is  $zzzz$   
 The Surd Quantity  $\sqrt{b}$  without the radical Sign is  $b$   
 These multiplied, the Product is the Square required  $zzzzb$

	<i>Exam. 5.</i>	<i>Exam. 6.</i>	<i>Exam. 7.</i>	<i>Exam. 8.</i>
Raise to the Square	$an\sqrt{p}$	$dz\sqrt{yx}$	$p\sqrt{xy}$	$da\sqrt{z}$
The Square	$aannp$	$ddzzyx$	$ppxy$	$ddaaaz$

*Exam. 5.* The rational Quantity  $an$  squared is  $aann$   
 The Surd Quantity  $\sqrt{p}$  without the radical Sign is  $p$   
 These multiplied, the Product is the Square required  $aannp$

*Exam. 6.* The rational Quantity  $dz$  squared is  $ddzz$   
 The Surd Quantity  $\sqrt{yx}$  without the radical Sign is  $yx$   
 These multiplied, the Product is the Square required  $ddzzyx$

*Exam. 7.* The rational Quantity  $p$  squared is  $pp$   
 The Surd Quantity  $\sqrt{xy}$  without the radical Sign is  $xy$   
 These multiplied, the Product is the Square required  $ppxy$

*Exam. 8.* The rational Quantity  $da$  squared is  $ddaa$   
 The Surd Quantity  $\sqrt{z}$  without the radical Sign is  $z$   
 These multiplied, the Product is the Square required  $ddaaaz$

Raise to the Square	$m\sqrt{pz}$	$mn\sqrt{d}$	$a\sqrt{rd}$	$py\sqrt{m}$
The Square	$mmpz$	$mmnnd$	$aard$	$ppyyym$

Raise to the Square	$x\sqrt{pd}$	$xn\sqrt{a}$	$z\sqrt{px}$	$az\sqrt{d}$
The Square	$xxpd$	$xxnna$	$zzpx$	$aaazd$

And if there are more Quantities than one under the radical Sign, connected with the Signs  $+$  or  $-$ , then after the rational Quantities are involved, or raised as high as is the Index of the

Surd; place these under the radical Quantities, without their Sign, then multiply them by the Rule of Multiplication at Art. 10. &c. and this will be the Square required.

	<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>
Raise to the Square	$a\sqrt{m+y}$	$b\sqrt{d+z}$	$m\sqrt{z-x}$
The Square is	$aam+aaay$	$bbd+bbbz$	$mmz-mmxx$

*Exam. 1.* The Surd Quantity  $\sqrt{m+y}$  } . . .  $m+y$   
 without the radical Sign is  
 The rational Quantity  $a$  squared is . . .  $aa$   
 These being multiplied according to Art. 10. }  $aam+aaay$   
 the Product is the Square required

*Exam. 2.* The Surd Quantity  $\sqrt{d+z}$  } . . .  $d+z$   
 without the radical Sign is  
 The rational Quantity  $b$  squared is . . .  $bb$   
 These multiplied, the Prod. is the Square required  $bbd+bbbz$

*Exam. 3.* The Surd Quantity  $\sqrt{z-x}$  } . . .  $z-x$   
 without the radical Sign is  
 The rational Quantity  $m$  squared is . . .  $mm$   
 These multiplied, the Product is the }  $mmz-mmxx$   
 Square required

	<i>Exam. 4.</i>	<i>Exam. 5.</i>	<i>Exam. 6.</i>
Raise to the Square	$z\sqrt{a+n}$	$x\sqrt{b-d}$	$d\sqrt{z+y}$
The Square	$zza+zzn$	$xxb-xxd$	$ddz+ddy$

*Exam. 4.* The Surd Quantity  $\sqrt{a+n}$  } . . .  $a+n$   
 without the radical Sign is  
 The rational Quantity  $z$  squared is . . .  $zz$   
 These multiplied, the Product is the }  $zza+zzn$   
 Square required

*Exam. 5.* The Surd Quantity  $\sqrt{b-d}$  } . . .  $b-d$   
 without the radical Sign is  
 The rational Quantity  $x$  squared is . . .  $xx$   
 These multiplied, the Product is the }  $xxb-xxd$   
 Square required

*Exam.*

*Exam. 6.* The Surd Quantity  $\sqrt{z+y}$  } - -  $z+y$   
 without the radical Sign is  
 The rational Quantity  $d$  squared is - -  $dd$   
 These multiplied, the Product is the } -  $ddz+ddy$   
 Square required

Raise to the Square  $y\sqrt{a-n}$   $n\sqrt{a+d}$   $d\sqrt{p-z}$   
 The Square  $yya-yy n$   $nna+nn d$   $ddp-dd z$

Raise to the Square  $e\sqrt{p-r}$   $d\sqrt{e+y}$   $z\sqrt{n-y}$   
 The Square  $ee p-ee r$   $dde+ddy$   $zz n-zzy$

45. *Case 3.* But if there are rational Quantities connected with the Surd Quantities by the Signs  $+$  or  $-$ , they are involved in the same Manner as compound Quantities, at Art. 32. carefully observing the Directions concerning the Multiplication of Surd Quantities, at Art. 40. and their Involution at Art. 43.

To raise to the Square or second Power -  $a+\sqrt{b}$   
 Putting down again the same Quantity -  $a+\sqrt{b}$

Now multiply  $a+\sqrt{b}$  by  $a$ , and  $a$  multiplied by  $a$ , the Product is  $aa$ , and  $\sqrt{b}$  multiplied by  $a$ , the Product is  $a\sqrt{b}$ , by Art. 40. therefore  $a+\sqrt{b}$  multiplied by  $a$  is }  $aa+a\sqrt{b}$

Then multiply  $a+\sqrt{b}$ , by  $\sqrt{b}$ , and  $a$  multiplied by  $\sqrt{b}$ , the Prod. is  $a\sqrt{b}$  by Art. 40. and  $\sqrt{b}$  multip. by  $\sqrt{b}$ , the Prod. is  $b$ , by Art. 43. whence  $a+\sqrt{b}$  multiplied by  $\sqrt{b}$  is }  $a\sqrt{b}+b$

The Sum is  $aa+2a\sqrt{b}+b$ : for  $a\sqrt{b}$  added to  $a\sqrt{b}$  is  $2a\sqrt{b}$ , by Art. 35. whence }  $aa+2a\sqrt{b}+b$   
 the Square of  $a+\sqrt{b}$  is

To raise to the Square or second Power -  $d+\sqrt{z}$   
 Putting down again the same Quantity -  $d+\sqrt{z}$

The Product from multiplying  $d+\sqrt{z}$  by  $d$ , }  $dd+d\sqrt{z}$   
 by what is mentioned in the last Example is

The Product from multiplying  $d+\sqrt{z}$  by }  $d\sqrt{z}+z$   
 $\sqrt{z}$ , by what is mentioned in the last Example is

The Sum added as in the last Example is }  $dd+2d\sqrt{z}+z$   
 the Square of  $d+\sqrt{z}$

To raise to the Square or second Power -  $x - \sqrt{a}$   
 Putting down again the same Quantity -  $x - \sqrt{a}$

The Prod. from multiplying  $x - \sqrt{a}$  by  $x$ , for  
 $x$  multiplied by  $x$ , the Prod. is  $xx$ , and  $-\sqrt{a}$   
 multiplied by  $x$ , the Prod. is  $-x\sqrt{a}$ , by Art. }  $xx - x\sqrt{a}$   
 40. the Signs of  $-\sqrt{a}$  and  $x$  being different

The Product from multiplying  $x - \sqrt{a}$  by  
 $-\sqrt{a}$ , for  $x$  multiplied by  $-\sqrt{a}$ , the Product  
 is  $-x\sqrt{a}$ , the Signs being unlike, but  $-\sqrt{a}$   
 multiplied by  $-\sqrt{a}$ , the Signs being alike, the  
 Product is  $\sqrt{a}a$  or  $a$  by Art. 39 and 43. }  $-x\sqrt{a} + a$

Their Sum is the Square of  $x - \sqrt{a}$  -  $xx - 2x\sqrt{a} + a$

To raise to the Square or second Power -  $y - \sqrt{x}$   
 Putting down again the same Quantity -  $y - \sqrt{x}$

The Product from multiplying  $y - \sqrt{x}$  by  $y$ , }  $yy - y\sqrt{x}$   
 from what is mentioned in the last Example is

The Product from multiplying  $y - \sqrt{x}$  by }  $-y\sqrt{x} + x$   
 $-\sqrt{x}$ , from what is said in the last Example is

Their Sum is the Square of  $y - \sqrt{x}$  -  $yy - 2y\sqrt{x} + x$

To raise to the Square or second Power -  $b + \sqrt{xa}$   
 Putting down the same Quantity -  $b + \sqrt{xa}$

Multiplying  $b + \sqrt{xa}$  by  $b$  we have -  $bb + b\sqrt{xa}$

Multiplying  $b + \sqrt{xa}$  by  $\sqrt{xa}$  we have  $b\sqrt{xa} + xa$

The Sum being the Square of  $b + \sqrt{xa}$ ,  $bb + 2b\sqrt{xa} + xa$

To raise to the Square or second Power -  $m + \sqrt{dz}$   
 Putting down the same Quantity -  $m + \sqrt{dz}$

Multiplying  $m + \sqrt{dz}$  by  $m$  we have -  $mm + m\sqrt{dz}$

Multiplying  $m + \sqrt{dz}$  by  $\sqrt{dz}$  we have  $m\sqrt{dz} + dz$

The Sum being the Square of  $m + \sqrt{dz}$ ,  $mm + 2m\sqrt{dz} + dz$

To raise to the Square or second Power -  $z - \sqrt{dn}$   
 Putting down the same Quantity -  $z - \sqrt{dn}$

$zz - z\sqrt{dn}$

$-z\sqrt{dn} + dn$

The Square of  $z - \sqrt{dn}$  -  $zz - 2z\sqrt{dn} + dn$

To raise to the Square or second Power	-	$p - \sqrt{yz}$
Putting down again the same Quantity	-	$p - \sqrt{yz}$
		<hr style="width: 100%;"/>
		$pp - p\sqrt{yz}$
		$- p\sqrt{yz} + yz$
		<hr style="width: 100%;"/>
The Square of $p - \sqrt{yz}$	-	$pp - 2p\sqrt{yz} + yz$

## OF EQUATIONS.

**H**AVING thus copiously explained all the Rules necessary to be known, in order to the Solution of Questions, we come now to their Use and Application in the Reduction of Equations, or the Method by which Problems are solved, and Questions answered.

When any Problem or Question is proposed to be answered *Algebraically*, for the several Numbers that are in the Question we generally put Letters, representing likewise the Numbers which are to be found by Letters, and for Distinction Sake use the *Vowels* for the unknown Numbers, or those that are to be found, and *Consonants* for those that are known, or given.

Then we begin to express all the Conditions of the Question, by ranging and connecting the Letters, by Help of the foregoing Signs, in such a Manner that they shall represent all the Circumstances of the Question, this being only to translate the Question from *English* into *Algebra*.

Thus if the Proposition, that 6 being added to 5, the Sum is equal to 11, was to be expressed in *Algebra*.

Now suppose  $b = 6$ ,  $d = 5$ ,  $m = 11$ .

Then the above Proposition will be expressed thus,  $b + d = m$ .

And when any Letters or Numbers are so connected, that between any of them there appears this Sign  $=$ , it is called an Equation, for the Sign  $=$  signifies *Equality* or *Equation*, and in the due ordering and managing these Equations consists the whole of the *Analytic Science*, or *Algebra*.

Equations consist of Quantities or Letters, some known, and others unknown, and the grand Work is so to manage the Equations, that express what is given in the Question, by the Rules of Certainty and Science, that all the known Quantities may at last be found on one Side of the Equation, and the unknown Quantity by itself on the other of the Equation: For when this is done, the Equation is brought to a Solution, and the Question is answered.

And

And that Part of *Algebra* which teaches how to manage these Quantities, so as to carry all the known Quantities on one Side, leaving the unknown Quantity by itself on the other Side of the Equation, is called the *Reduction of Equations*, which is done by *Addition*, *Subtraction*, *Multiplication*, *Division*, *Involution*, and *Evolution*, according as the Case requires.

*To reduce an Equation by Addition, or Subtraction.*

46. **W**HEN any known Quantities are on the same Side of the Equation with the unknown Quantity, and connected by the Signs  $+$  or  $-$ , to reduce such an Equation is only to *transpose* or *place the known Quantities on the other Side of the Equation, or Sign of Equality, prefixing to them their contrary Sign*, that is, *those Quantities which have the Sign  $+$ , after they are transposed must have the Sign  $-$ , and those which have the Sign  $-$  must have the Sign  $+$ .*

Question 1. *To find that Number to which 6 being added, and subtracting 15 from this Sum, the Remainder may be equal to 11.*

Now suppose  $a =$  the Number sought,  $b = 6, d = 15, m = 11$ .

Then I am to find a Number, which call	1	$a$
To which 6 or $b$ being added, it is by Art. 6.	2	$a + b$
From which Sum 15 or $d$ is to be sub-	3	$a + b - d$
tracted, that is, to $a + b$ connect $d$ by		
the Sign $-$ , then it is	4	$a + b - d = m$
Which $a + b - d$ is to be equal to 11 or		
$m$ , that is		
Now to reduce this Equation, or to an-	5	$a + b = m + d$
swer the Question, I observe $d$ , a known		
Quantity, is on the same Side of the		
Equation with the unknown Quantity		
$a$ , therefore transpose $d$ , that is, write	6	$a = m + d - b$
down the remaining Part of that Side of		
the Equation without $d$ , and place it on		
the other Side with the Sign $+$ , it hav-		
ing before the Sign $-$ , then we have		
Again $b$ is a known Quantity on the same	6	$a = m + d - b$
Side of the Equation with $a$ , then by		
taking it away from that Side of the		
Equation, and placing it on the other Side		
with the contrary Sign, or $-$ , we have		

Here



Here the Question is solved, for the unknown Number or Quantity  $a$ , is equal to the Number represented by  $m$ , added to the Number represented by  $d$ , from which Sum subtracting the Number represented by  $b$ .

11 represented by  $m$

15 represented by  $d$

26 Sum of the Numbers represented by  $m$  and  $d$

6 represented by  $b$ , to be subtracted

Remains 20 which is  $a$ , or the Number sought.

And that this is the Number required, is thus proved, from the Conditions of the Question.

I say the Number sought is	—	—	20
For if to this is added	—	—	6
The Sum is	—	—	26
From which subtracting	—	—	15
There remains as the Question required	—	—	11

**Question 2.** *A Man being asked how many Shillings he had, said, if you add 15 to their Number, and then subtract 20 from that Sum, and then add 19 to the Remainder, I shall have 64 Shillings. How many Shillings had he?*

Let  $a$  = the Number of Shillings sought,  $b = 15$ ,  $d = 20$ ,  $m = 19$ ,  $n = 64$ .

Then, A Man had a certain Number of Shillings, which call	1	$a$
To which 15 or $b$ being added we have, by Art. 6.	2	$a + b$
From which Sum taking away 20 or $d$ , that is, connect $d$ by the Sign —	3	$a + b - d$
To which adding 19 or $m$ we have, by Art. 6.	4	$a + b - d + m$
Which $a + b - d + m$ is to be equal to 64 or $n$ , hence	5	$a + b - d + m = n$
Now to reduce this Equation, or answer the Question: I begin with transposing $m$ a known Quantity, by putting down the remaining Part of that Side of the Equation, and placing $m$ on the other Side with the contrary Sign, which gives	6	$a + b - d = n - m$

M

And

And to transpose  $d$  another known Quantity, put down the remaining Part of that Side of the Equation, and  $d$  on the other Side with a contrary Sign, whence we have

$$\left. \begin{array}{l} \text{And lastly, by transposing } b, \text{ that is,} \\ \text{placing it on the other Side of the} \\ \text{Equation with a contrary Sign, we} \\ \text{have} \end{array} \right\} \begin{array}{l} 7 \quad a + b = n - m + d \\ 8 \quad a = n - m + d - b \end{array}$$

That is, if from the Number represented by  $n$  we subtract that represented by  $m$ , and to the Remainder add the Number represented by  $d$ , and from this Sum subtract the Number represented by  $b$ , the Remainder will be the Number sought.

64 represented by  $n$   
 19 represented by  $m$ , to be subtracted  


---

 45 or  $n - m$   
 20 represented by  $d$ , to be added  


---

 65 or  $n - m + d$   
 15 represented by  $b$ , to be subtracted

50 the Number sought or  $a$ ; and therefore the Man had 50s. at first, which is thus proved, from the Conditions of the Question.

I say he had at first	—	—	50 s.
For if to them you add	—	—	15
			<hr/> 65
And from that Sum subtract	—	—	20
			<hr/> 45
And then add to the Remainder	—	—	19
It makes what the Question requires		—	<hr/> 64

Question 3. A Countryman asked another how many Eggs he had, Why, says he, if you subtract 15 from their Number, and then add 21 to those that are left, and subtract 7 from that Sum, but if you add 19 to what is then left I shall have 43 Eggs. How many Eggs had he?

Let

# To reduce an Equation, &c. 83

Let  $a$  = the Number of Eggs,  $b = 15$ ,  $d = 21$ ,  $m = 7$ ,  
 $n = 19$ ,  $p = 43$ .

Now the Countryman had a Num- ber of Eggs, which call -	1	$a$
From which 15 or $b$ being sub- stracted, or connecting $b$ by the Sign —, we have -	2	$a - b$
To which $a - b$ , if we add 21 or $d$ , we have by Art. 6. -	3	$a - b + d$
From which Sum subtracting 7, or connecting $m$ by the Sign —	4	$a - b + d - m$
To which adding 19 or $n$ , we have by Art. 6.	5	$a - b + d - m + n$
And this $a - b + d - m + n$ is to be equal to 43 or $p$ , hence	6	$a - b + d - m + n = p$
Now to reduce this Equation, or answer the Question, I begin with transposing $n$ , by putting down the remaining Part of that Side of the Equation, and $n$ on the other Side with its contrary Sign, then	7	$a - b + d - m = p - n$
Now transpose $m$ , by putting down the remaining Part of that Side of the Equation, and $m$ on the other Side with its contrary Sign, and we have	8	$a - b + d = p - n + m$
Then transpose $d$ , by putting down the remaining Part of that Side of the Equation, and $d$ on the other Side with its contrary Sign, then	9	$a - b = p - n + m - d$
Lastly, transpose $b$ , by putting down the remaining Part of that Side of the Equation, and $b$ on the other Side with its con- trary Sign, and it is	10	$a = p - n + m - d + b$

Hence  $a$ , the unknown Quantity or Number of Eggs, is  
equal to the Number represented by  $p$ , subtracting from it the  
Number represented by  $n$ , adding to this Remainder the Number  
represented by  $m$ , subtracting from this Sum the Number  
represented

represented by  $d$ , and adding to the Remainder the Number represented by  $b$ .

$$\begin{array}{r}
 43 \text{ represented by } p \\
 19 \text{ represented by } n, \text{ to be subtracted} \\
 \hline
 24 \text{ or } p - n \\
 7 \text{ represented by } m, \text{ to be added} \\
 \hline
 31 \text{ or } p - n + m \\
 21 \text{ represented by } d, \text{ to be subtracted} \\
 \hline
 10 \text{ or } p - n + m - d \\
 15 \text{ represented by } b, \text{ to be added} \\
 \hline
 \end{array}$$

25 the Number sought or  $a$ ; and therefore the Man had 25 Eggs, which is thus proved, from the Conditions of the Question.

I say the Man had	-	-	25 Eggs
For if from them you subtract	-	-	19
			<hr/> 10
And to the Remainder add	-	-	21
			<hr/> 31
And from this Sum subtract	-	-	7
			<hr/> 24
And to the Remainder add	-	-	19
It makes what the Question requires	-	-	<hr/> 43

Question 4. To find that Number to which 19 being added, if from that Sum we subtract 50, and add 7 to the Remainder, and subtract 60 from this Sum, and by adding 6 to that Remainder, this Sum may be 22.

Let  $a$  = Number sought,  $b = 19$ ,  $d = 50$ ,  $m = 7$ ,  $n = 60$ ,  $p = 6$ ,  $g = 22$ .

$$\begin{array}{l}
 \text{Now I am to find a Number, } \left. \begin{array}{l} \text{which call} \\ \text{To which 19 or } b \text{ being added,} \\ \text{we have by Art. 6.} \end{array} \right\} \begin{array}{l} 1 \\ 2 \end{array} \left| \begin{array}{l} a \\ a + b \end{array} \right. \\
 \text{From which Sum subtracting 50} \\
 \text{or } d, \text{ that is, connecting } d \text{ by} \\
 \text{the Sign } -, \text{ and it is} \end{array} \left. \begin{array}{l} \\ \\ \end{array} \right\} 3 \left| \begin{array}{l} a + b - d \end{array} \right.$$

And

And to this Remainder adding 7 or $m$ , we have by Art. 6.	4	$a + b - d + m$
From this Sum subtracting 60 or $n$ , that is, connecting $n$ by the Sign —	5	$a + b - d + m - n$
And to this Remainder adding 6 or $p$ , we have	6	$a + b - d + m - n + p$
And this $a + b - d + m - n + p$ is to be equal to 22 or $g$ , hence	7	$a + b - d + m - n + p = g$
Now to answer the Question, transpose $p$ , by putting down the remaining Part of that Side of the Equation, and $p$ on the other Side with its con- trary Sign, hence	8	$a + b - d + m - n = g - p$
Then transpose $n$ , by putting down the remaining Part of that Side of the Equation, and $n$ on the other Side with its contrary Sign, then	9	$a + b - d + m = g - p + n$
Then transpose $m$ , by putting down the remaining Part of that Side of the Equation, and $m$ on the other Side with its contrary Sign, whence	10	$a + b - d = g - p + n - m$
Then transpose $d$ , by putting down the remaining Part of that Side of the Equation, and $d$ on the other Side with its contrary Sign, and	11	$a + b = g - p + n - m + d$
Lastly, transpose $b$ , by putting down the remaining Part of that Side of the Equation, and $b$ on the other Side with its contrary Sign, we have	12	$a = g - p + n - m + d - b$

Hence  $a$ , the unknown Number, is equal to the Number represented by  $g$ , subtracting from it the Number represented by  $p$ , adding to the Remainder the Number represented by  $n$ , subtracting from this Sum the Number represented by  $m$ , adding to the Remainder the Number represented by  $d$ , and subtracting from this Sum the Number represented by  $b$ .

22 the Number represented by  $g$   
 6 the Number represented by  $p$ , subtract

16 or  $g - p$

60 the Number represented by  $n$ , add

76 or  $g - p + n$

7 the Number represented by  $m$ , subtract

69 or  $g - p + n - m$

50 the Number represented by  $d$ , add

119 or  $g - p + n - m + d$

19 the Number represented by  $b$ , subtract

100 the Number sought or  $a$ , which is thus proved, from  
 the Conditions of the Question.

I say the Number sought was	-	-	100
For if to that you add	-	-	<u>19</u>
			119
And from the Sum subtract	-	-	<u>50</u>
			69
And to the Remainder add	-	-	<u>7</u>
			76
And from the Sum subtract	-	-	<u>60</u>
			16
And add to the Remainder	-	-	<u>6</u>
It makes what the Question requires	-	-	22

The Directions to the two following Questions are not quite so copious, that the Judgment of the Learner may be a little more exercised.

Question 5. *A Number of Men were walking on a Bowling-Green, one Man asked another how many there were, the other replied, if you subtract 7 from their Number, and add 15 to the Remainder, and subtract 9 from the Sum, and add 56 to the Remainder, and subtract 2 from that Sum, this will leave 100. To find the Number of Men on the Bowling-Green.*

Let  $a$  = Number of Men on the Bowling-Green,  $b = 7$ ,  
 $d = 15$ ,  $g = 9$ ,  $m = 56$ ,  $n = 2$ ,  $p = 100$ .

I am

I am to find the Number of Men on the Bowling-Green, which call - - -	1	$a$
From which 7 or $b$ being subtracted, which is only to connect $b$ by the Sign —	2	$a - b$
To which Remainder adding 15 or $d$ , we have by Art. 6.	3	$a - b + d$
From which Sum subtracting 9 or $g$ , or connecting $g$ by the Sign —, and we have	4	$a - b + d - g$
To this Remainder adding $m$ or 56, by Art. 6.	5	$a - b + d - g + m$
From which Sum subtracting 2 or $n$ , that is, connecting $n$ by the Sign —, it is	6	$a - b + d - g + m - n$
Which $a - b + d - g + m - n$ is to be equal to 100 or $p$ , hence - - -	7	$a - b + d - g + m - n = p$
By transposing $n$ we have -	8	$a - b + d - g + m = p + n$
By transposing $m$ we have -	9	$a - b + d - g = p + n - m$
By transposing $g$ we have -	10	$a - b + d = p + n - m + g$
By transposing $d$ we have -	11	$a - b = p + n - m + g - d$
By transposing $b$ we have -	12	$a = p + n - m + g - d + b$

100 is the Number represented by  $p$

2 or  $n$ , to be added

102 or  $p + n$

56 or  $m$ , to be subtracted

46 or  $p + n - m$

9 or  $g$ , to be added

55 or  $p + n - m + g$

15 or  $d$ , to be subtracted

40 or  $p + n - m + g - d$

7 or  $b$ , to be added

47 the Number sought or  $a$ , for  $a = p + n - m + g - d + b$ .

Now to prove 47 was the Number of Men that were on the Bowling-Green, let us try if it will answer the Conditions of the Question.

I say

I say the Number of Men were	-	-	47
For if from them you subtract	-	-	7
			<hr/> 40
And add to the Remainder	-		15
			<hr/> 55
And from the Sum subtract	-	-	9
			<hr/> 46
And add to the Remainder	-	-	56
			<hr/> 102
And from the Sum subtract	-	-	2
It makes what the Question requires	-		<hr/> 100

Question 6. *A Person required another to tell him how many Shillings he had, by saying that if to their Number was added 5, and from this Sum subtracting 3, and adding 16 to the Remainder, and from that Sum subtracting 50, and adding 54 to the Remainder, he should then have 43 Shillings. How many Shillings had he?*

Let  $a$  = the Number of Shillings sought,  $b = 5$ ,  $d = 3$ ,  $m = 16$ ,  $n = 50$ ,  $p = 54$ ,  $q = 43$ .

The Person had a certain Number of Shillings, which call	}	1	$a$
To which 5 or $b$ being added, we have	}	2	$a + b$
From which Sum subtracting 3 or $d$ , we have	}	3	$a + b - d$
To which Remainder adding 16 or $m$ , we have	}	4	$a + b - d + m$
From which Sum subtracting 50 or $n$ , we have	}	5	$a + b - d + m - n$
To which Remainder adding 54 or $p$ , we have	}	6	$a + b - d + m - n + p$
Which $a + b - d + m - n + p$ is to be equal to 43 or $q$ , hence	}	7	$a + b - d + m - n + p = q$
The Question being now expressed in Algebra, by supposing $p$ , we have	}	8	$a + b - d + m - n = q - p$

By



# To reduce an Equation, &c. 89

By transposing  $n$  we have -  $\left| \begin{array}{l} 9 \\ 10 \\ 11 \\ 12 \end{array} \right| \begin{array}{l} a + b - d + m = q - p + n \\ a + b - d = q - p + n - m \\ a + b = q - p + n - m + d \\ a = q - p + n - m + d - b \end{array}$

43 is the Number represented by  $q$   
—54 from which subtracting 54 or  $p$ , there remains — 11  
 —11 or  $q - p$ , see below.\*  
50 adding 50 or  $n$  to — 11, the Sum is 39  
39 or  $q - p + n$   
16 from which subtracting 16 or  $m$   
23 or  $q - p + n - m$   
3 to which adding 3 or  $d$   
26 or  $q - p + n - m + d$   
5 from which subtracting 5 or  $b$   
21 hence 21 is the Number sought ; which is thus proved :

I say the Person had	-	-	21 Shillings
For if to them you add	-	-	5
			<u>26</u>
And from the Sum subtract	-	-	3
			<u>23</u>
And to the Remainder add	-	-	16
			<u>39</u>
And from the Sum subtract	-	-	50
There remains a <i>negative</i> or	-	-	<u>—11</u>
And if to this Remainder we add	-	-	54
It makes what the Question requires	-	-	<u>43</u>

\* When a *negative* Number is to be subtracted from an *affirmative* Number, and the negative Number is greatest, as in this Case, it is only to take the Difference of the two Numbers, and place the Sign — before it; and if the next Number to be added is *affirmative*, and greater than the *negative* Remainder, then it is only subtracting the *negative* Remainder from the *affirmative* Number which is to be added, and this will be the Sum.

If the Learner finds any Difficulty in conceiving this, he may collect all the *affirmative* Numbers into one Sum, and all the *negative* Numbers into another; and subtracting the Sum of the *Negatives* from the Sum of the *Affirmatives*, the Remainder is the Answer to the Question.

In the last Question,

The <i>affirmative</i> Quantities } or Numbers are - - -	$q = 43$	
	$n = 50$	
	$d = 3$	
	96	
Sum of the negative Numbers	— 75	
	21 = a as before	

The <i>negative</i> Quantities } or Numbers are - - -	$-p = -54$	
	$-m = -16$	
	$-b = -5$	
	— 75	

### *To reduce an Equation by Multiplication.*

47. In the last Article, the unknown Quantity was connected with the known Quantities by the Signs  $+$  or  $-$  only, but it may happen that the unknown Quantity may be *divided* by some known Quantity; in this Case, *multiply every Part or all the Terms of the Equation by that known Quantity*; and the Part of the Equation containing the unknown Quantity will be then multiplied and divided by the same Quantity, take down this Equation, rejecting the known Quantity from that Part of the Equation where it both multiplies and divides the unknown Quantity, by Art. 20. it being in both Dividend and Divisor: After this Equation is set down, if there are any other Quantities connected with the unknown one by the Signs  $+$  or  $-$ , transpose them to the other Side of the Equation as in the last Article, by which Method we shall have all the known Quantities on one Side of the Equation, and the unknown one by itself on the other Side, which is the Solution of the Question.

Question

**Question 7.** *A Gamester challenging another to play for as many Guineas as he had in his Hand, the other required to know how many there were, he replied, if you divide their Number by 5, and add 19 to the Quotient, I shall then have 23 Guineas in my Hand. How many Guineas had he ?*

Let  $a$  = the Number of Guineas fought,  $b = 5$ ,  $d = 19$ ,  $m = 23$ .

Then the Gamester had a certain Num- ber of Guineas, which call -	} 1	$a$
Which being divided by 5 or $b$ , the Quo- tient is by Art. 27. -	} 2	$\frac{a}{b}$
To which Quotient $\frac{a}{b}$ if we add 19 or $d$ , we have by Art. 6. -	} 3	$\frac{a}{b} + d$
And this $\frac{a}{b} + d$ , is to be equal to 23 or $m$ , therefore we have -	} 4	$\frac{a}{b} + d = m$
The Question being expressed in <i>Algebra</i> by the Equation $\frac{a}{b} + d = m$ , in which the unknown Quantity $a$ being divided by $b$ ; now by the Rule, multiply every Part or Quantity in the Equa- tion by $b$ , and in this Multiplication, multiply only the Numerator $a$ of the Quantity $\frac{a}{b}$ by $b$ , according to the Rule of Vulgar Fractions in Arithmetic, and we have -	} 5	$\frac{ab}{b} + bd = bm$
Because $b$ is in both Dividend and Divisor of the Quantity $\frac{ab}{b}$ , hence by the Rule, rejecting $b$ from $\frac{ab}{b}$ only, and placing down the remaining Part $a$ , and all the other Parts of the Equation, without any Alteration, we have	} 6	$a + bd = bm$
Transposing $bd$ by the last Article, it being a known Quantity, then -	} 7	$a = bm - bd$

Here the Question is answered, for  $a$  the unknown Quantity is equal to the Product of the two Numbers represented by  $b$  and  $m$ , subtracting from it the Product of the two Numbers represented by  $b$  and  $d$ .

The Number represented by  $b$  is 5, the Number represented by  $m$  is 23, which two Numbers being multiplied is  $b m$  or  $\left. \begin{array}{l} \text{The Number represented by } b \text{ is 5, the Number represented by } m \text{ is 23, which two Numbers being multiplied is } b m \text{ or} \end{array} \right\} 115$

The Number represented by  $b$  is 5, the Number represented by  $d$  is 19, which two Numbers being multiplied is  $b d$  or  $\left. \begin{array}{l} \text{The Number represented by } b \text{ is 5, the Number represented by } d \text{ is 19, which two Numbers being multiplied is } b d \text{ or} \end{array} \right\} 95$

Subtracting  $b d$  from  $b m$ , that is, 95 from 115, leaves  $\left. \begin{array}{l} \text{Subtracting } b d \text{ from } b m, \text{ that is, 95 from 115, leaves} \\ b m - b d \text{ or} \end{array} \right\} 20$

Which is the Number sought, or the Guineas the Gamester had, and is proved from the Conditions of the Question, thus,

I say the Gamester had  $\quad \quad \quad 20$  Guineas

For if that Number is divided by 5, the Quotient is  $\frac{4}{\quad}$

But if to this 4 we add  $\quad \quad \quad 19$

It makes what the Question requires  $\quad \quad \quad 23$

Question 8. To find that Number which being divided by 15, if to the Quotient we add 27, and subtract 13 from the Sum, the Remainder will be 18.

Let  $a$  = the Number sought,  $b = 15$ ,  $d = 27$ ,  $m = 13$ ,  $p = 18$ .

Now I am to find a Number, which call $\quad \quad \quad$	$\left. \begin{array}{l} \text{Now I am to find a Number, which call} \\ \text{Which being divided by 15 or } b, \text{ we have by Art. 27.} \end{array} \right\}$	$\left. \begin{array}{l} 1 \quad a \\ 2 \quad \frac{a}{b} \end{array} \right\}$
To the Quotient or $\frac{a}{b}$ , if we add 27 or $d$ , we have by Art. 6.	$\left. \begin{array}{l} \text{To the Quotient or } \frac{a}{b}, \text{ if we add 27 or } d, \text{ we have by Art. 6.} \end{array} \right\}$	$3 \quad \frac{a}{b} + d$
From this Sum if we subtract 13 or $m$ , that is, connect $m$ by the Sign $-$ , it is $\quad \quad \quad$	$\left. \begin{array}{l} \text{From this Sum if we subtract 13 or } m, \text{ that is, connect } m \text{ by the Sign } -, \text{ it is} \end{array} \right\}$	$4 \quad \frac{a}{b} + d - m$
Which $\frac{a}{b} + d - m$ is by the Question to be equal to 18 or $p$ , hence we have $\quad \quad \quad$	$\left. \begin{array}{l} \text{Which } \frac{a}{b} + d - m \text{ is by the Question to be equal to 18 or } p, \text{ hence we have} \end{array} \right\}$	$5 \quad \frac{a}{b} + d - m = p$

The

The Question being now expressed in *Algebra* by this Equation  $\frac{a}{b} + d - m = p$ , and

the unknown Quantity  $a$  being divided by  $b$ , multiply every Part of the Equation by  $b$  as in the last Question, and then we have

$$6 \quad \frac{ab}{b} + db - mb = pb$$

Because  $b$  is in both Dividend and Divisor of the Quantity  $\frac{ab}{b}$ , reject  $b$  from this Quantity only as in the last Question,

placing down  $a$  and the remaining Quantities in the Equation without any Alteration, then we have

$$7 \quad a + db - mb = pb$$

Because  $mb$  is a known Quantity, transpose it by the Directions in the last Article, and we have

$$8 \quad a + db = pb + mb$$

Because  $db$  is a known Quantity, transpose it by the same Directions, and we have

$$9 \quad a = pb + mb - db$$

Now  $a$  the unknown Quantity being by itself on one Side of the Equation, the Question is solved; for  $a$ , the unknown Quantity, is equal to the Product of the two Numbers represented by  $p$  and  $b$ , added to the Product of the two Numbers represented by  $m$  and  $b$ , subtracting from this Sum the Product of the Numbers represented by the Letters  $d$  and  $b$ .

The Number represented by  $p$  is 18, the Number represented by  $b$  is 15, the Product of these two Numbers is  $pb$  or

270

The Number represented by  $m$  is 13, the Number represented by  $b$  is 15, the Product of these two Numbers is  $mb$  or

195

The Sum is  $pb + mb$  or

465

The

The Number represented by  $d$  is 27, the Number represented by  $b$  is 15, the Product of these two Numbers is 405, which being subtracted from the Sum of the other two Products

$$\text{Leaves } pb - mb - db \text{ or } a \quad \quad \quad \underline{60}$$

Therefore 60 is equal to  $a$ , or 60 is the Number sought, which is thus proved from the Question.

$$\begin{array}{rcl} \text{I say the Number sought is} & - & 60 \\ \text{For if that is divided by 15 the Quotient is} & - & 4 \\ \text{To which Quotient, or 4, if we add} & - & 27 \\ \text{The Sum is} & - & 31 \\ \text{And if from this Sum we subtract} & - & 13 \\ \text{There remains what the Question requires} & - & 18 \end{array}$$

Question 9. *A Man being asked how many Shillings he had, replied, if you divide the Number I have by 25, and subtract 3 from the Quotient, and then add 51 to this Remainder, and from the Sum subtracting 40, I shall have 12 Shillings left. How many Shillings had he?*

Let  $a$  = the Number of Shillings the Man had,  $b = 25$ ,  $d = 3$ ,  $m = 51$ ,  $p = 40$ ,  $z = 12$ .

$$\begin{array}{lcl} \text{Now the Man had a certain} & \left. \begin{array}{l} \text{Number of Shillings, which} \\ \text{call} \end{array} \right\} & 1 \quad a \\ \text{Which being divided by 25 or} & \left. \begin{array}{l} b, \text{ we have by Art. 27.} \end{array} \right\} & 2 \quad \frac{a}{b} \\ \text{From the Quotient or } \frac{a}{b}, \text{ if} & \left. \begin{array}{l} \text{we subtract 3 or } d, \text{ that is,} \\ \text{connecting } d \text{ by the Sign } - \end{array} \right\} & 3 \quad \frac{a}{b} - d \\ \text{To the Remainder adding 51 or } m, & \left. \begin{array}{l} \text{we have by Art. 6.} \end{array} \right\} & 4 \quad \frac{a}{b} - d + m \\ \text{From which subtracting 40 or } p, & \left. \begin{array}{l} \text{that is, connecting } p \text{ by} \\ \text{the Sign } -, \text{ we have} \end{array} \right\} & 5 \quad \frac{a}{b} - d + m - p \\ \text{Which } \frac{a}{b} - d + m - p \text{ is by} & \left. \begin{array}{l} \text{the Question, to be equal to} \\ 12 \text{ or } z, \text{ hence we have} \end{array} \right\} & 6 \quad \frac{a}{b} - d + m - p = z \end{array}$$

The Question being now expressed in <i>Algebra</i> , and the unknown Quantity $a$ being divided by $b$ , multiply every Quantity in the Equation by $b$ , as in the two last Questions, then we have	7 $\frac{ab}{b} - db + mb - pb = zb$
And rejecting $b$ out of the Quantity $\frac{ab}{b}$ only, because it is in both Dividend and Divisor, and setting down the rest as in the two last Questions, we have	8 $a - db + mb - pb = zb$
Because $pb$ is a known Quantity, transpose it by Art. 46. and we have	9 $a - db + mb = zb + pb$
Because $mb$ is a known Quantity, transpose it in like Manner, then we have	10 $a - db = zb + pb - mb$
Because $db$ is a known Quantity, by transposing it we have	11 $a = zb + pb - mb + db$

Now it appears the unknown Quantity, or  $a$ , is equal to the Product of the two Numbers represented by  $z$  and  $b$ , added to the Product of the two Numbers represented by  $p$  and  $b$ , subtracting from this Sum the Product of the two Numbers represented by  $m$  and  $b$ , and adding to this Remainder the Product of the two Numbers represented by  $d$  and  $b$ .

The Number represented by  $z$  is 12, and that by  $b$  }  
is 25, the Product of these two is  $zb$  or - } 300

The Number represented by  $p$  is 40, and that by  $b$  }  
is 25, the Product of these two is  $pb$  or - } 1000

The Sum is  $zb + pb$  or - - - 1300

The Number represented by  $m$  is 51, and that by  $b$  }  
is 25, the Product of these two is  $mb$  or - } 1275

Which being subtracted from the Sum of the other }  
two, leaves  $zb + pb - mb$  or - - - } 25

The Number represented by  $d$  is 3, and that by  $b$  }  
is 25, the Product of these two is  $db$  or - } 75

Which added to the last Remainder, the Sum is }  
 $zb + pb - mb + db$  or  $a$  } 100

Whence

Whence the unknown Quantity  $a$ , or the Number of Shillings the Man had is 100, which is thus proved, from the Conditions of the Question.

	Shillings.
I say the Man had	100
For if that Number is divided by 25, the Quotient is	4
From which Quotient if we subtract	3
Remains	1
To which adding	51
The Sum is	52
From which subtracting	40
There remains what the Question requires	12

Question 10. *A Country Servant, who understood Algebra, being asked by his Master how many Cows there were in the Field, replied, if you add 13 to their Number, and divide that Sum by 8, and then add 19 to the Quotient, and subtract 11 from that Sum, there will be 12 Cows left. How many Cows were there?*

Let  $a$  = the Number of Cows,  $b = 13$ ,  $d = 8$ ,  $m = 19$ ,  $p = 11$ ,  $x = 12$ .

Now there were in the Field a certain Number of Cows, which call	1	$a$
To which 13 or $b$ being added, we have by Art. 6.	2	$a + b$
Which $a + b$ being divided by 8 or $d$ , we have by Art. 28.	3	$\frac{a + b}{d}$
To which if we add 19 or $m$ , we have by Art. 6.	4	$\frac{a + b}{d} + m$
From which if we subtract 11 or $p$ , we have by connecting $p$ with the Sign —	5	$\frac{a + b}{d} + m - p$
Which $\frac{a + b}{d} + m - p$ is by the Question to be equal to 12 or $x$ , hence we have	6	$\frac{a + b}{d} + m - p = x$

Because



Because  $a$ , the unknown Term,  
is Part of the Fraction  $\frac{a+b}{d}$

where the Divisor is  $d$ , there-  
fore multiplying every Quan-  
tity in the Numerator of  
the Fraction by  $d$ , by the  
Rule of Vulgar Fractions in  
Arithmetick, and the other  
Quantities as before, then

Because  $d$  is in every Term of  
the Dividend and Divisor of  
the Fraction  $\frac{ad+bd}{d}$ , re-

ject  $d$ , from  $\frac{ad+bd}{d}$  only,

by Art. 22. and 24, and set  
down all the rest as before,  
then

Now begin to transpose  $pd$ , it  
being a known Quantity,  
then we have

Because  $md$  is a known Quan-  
tity, therefore transpose it,  
and we have

Because  $b$  is a known Quan-  
tity, therefore transpose it,  
and we have

$$7 \quad \frac{ad+bd}{d} + md - pd = xd$$

$$8 \quad a + b + md - pd = xd$$

$$9 \quad a + b + md = xd + pd$$

$$10 \quad a + b = xd + pd - md$$

$$11 \quad a = xd + pd - md - b$$

By this it appears that  $a$ , the unknown Quantity, is equal to the Product of the two Numbers represented by  $x$  and  $d$ , added to the Product of the two Numbers represented by  $p$  and  $d$ , subtracting from this Sum the Product of the two Numbers represented by  $m$  and  $d$ , subtracting still from this Remainder the Number represented by  $b$ .

The Product of the two Numbers represented by  $x$  } 96  
and  $d$  is  $xd$ , or

The Product of the two Numbers represented by  $p$  } 88  
and  $d$  is  $pd$ , or

Their Sum is  $xd + pd$ , or 184



From this Sum if we subtract 16 or $n$ , we have	5	$\frac{a-b}{c} + d - n$
To which Remainder if we add 68 or $p$ , we have	6	$\frac{a-b}{c} + d - n + p$
Which $\frac{a-b}{c} + d - n + p$ is, by the Question, to be equal to 101 or $x$ , hence	7	$\frac{a-b}{c} + d - n + p = x$
Because $a$ , the unknown Quantity, is Part of $\frac{a-b}{c}$ , which being di- vided by $c$ , therefore multiplying by $c$ , as in the last Question, we have	8	$\frac{ca-cb}{c} + cd - cn + cp = cx$
Because $c$ is in every Term of the Dividend and Di- visor of $\frac{ca-cb}{c}$ reject $c$ , as in the last Question, and set down all the rest as before, and we have	9	$a - b + cd - cn + cp = cx$
Now transpose $cp$ , it be- ing a known Quantity, then it is	10	$a - b + cd - cn = cx - cp$
Transpose $cn$ , it being a known Quantity, and we have	11	$a - b + cd = cx - cp + cn$
Transpose $cd$ , it being a known Quantity, and we have	12	$a - b = cx - cp + cn - cd$
And transposing $b$ , it being a known Quantity, we have	13	$a = cx - cp + cn - cd + b$

Hence it appears that  $a$ , the unknown Quantity, is equal to the Product of the two Numbers represented by  $c$  and  $x$ , subtracting from it the Product of the two Numbers represented by  $c$  and  $p$ , adding

adding to that Remainder the Product of the two Numbers represented by  $c$  and  $n$ , subtracting from this Sum the Product of the two Numbers represented by  $c$  and  $d$ , and adding to this Remainder the Number represented by  $b$ .

The Product of the two Numbers represented by $c$	5050
and $x$ is $cx$ , or	-
The Product of the two Numbers represented by $c$	3400
and $p$ is $cp$ , or	-
The Remainder is $cx - cp$ , or	1650
The Product of the two Numbers represented by $c$	800
and $n$ is $cn$ , or	-
Which added to the last Remainder, the Sum is	2450
$cx - cp + cn$ , or	-
The Product of the two Numbers represented by $c$	1950
and $d$ is $cd$ , or 1950, which being subtracted	-
The Remainder is $cx - cp + cn - cd$ , or	500
Adding the Number represented by $b$ , or	115
The Sum is $cx - cp + cn - cd + b$ , which is equal	615
to $a$ , the Number sought	-

And that there were 615 Men is proved from the Conditions of the Question.

	Men.
I say there were	615
For if from them we subtract	115
Remains	500
Which being divided by 50, the Quotient is	10
To which adding	39
The Sum is	49
From which subtracting	16
The Remainder is	33
To which adding	68
There remains what the Question requires	101

**Question 12.** *There is a certain Number to which 9 being added, and dividing this Sum by 5, if from this Quotient we subtract 6, and add 101 to the Remainder, from that Sum subtracting 10, there remains 97. What is the Number?*

Let

Let  $a$  = the Number sought,  $b = 9$ ,  $c = 5$ ,  $d = 6$ ,  $m = 101$ ,  
 $p = 10$ ,  $x = 97$ .

Now I am to find a certain } Number, which call - }	1	$a$
To which 9, or $b$ , being } added, we have by Art. 6. }	2	$a + b$
This being divided by 5, } or $c$ , we have by Ar- } ticle 28. - - }	3	$\frac{a + b}{c}$
From which subtracting 6, } or $d$ , we have - }	4	$\frac{a + b}{c} - d$
To which adding 101, or } $m$ , we have by Art. 6. }	5	$\frac{a + b}{c} - d + m$
From this subtracting 10, } or $p$ , that is, connecting } $p$ by the Sign -, it is }	6	$\frac{a + b}{c} - d + m - p$
Which $\frac{a + b}{c} - d + m - p$ } is to be equal to 97, or } $x$ , hence - - }	7	$\frac{a + b}{c} - d + m - p = x$
The Question being thus } expressed in <i>Algebra</i> , be- } gin and multiply by $c$ , } for the Reason in the last } Question, then we have }	8	$\frac{ca + cb}{c} - cd + cm - cp = cx$
Rejecting $c$ from $\frac{ca + cb}{c}$ , } and setting down the rest } as before, then - }	9	$a + b - cd + cm - cp = cx$
Now transposing $c p$ , it } being a known Quantity, } we have - - }	10	$a + b - cd + cm = cx + cp$
Transposing $cm$ , it being } a known Quantity, we } have - - }	11	$a + b - cd = cx + cp - cm$
Transposing $c d$ , it being } a known Quantity, we } have - - }	12	$a + b = cx + cp - cm + cd$
Lastly, transposing $b$ , it } being a known Quantity, } we have - - }	13	$a = cx + cp - cm + cd - b$



then you will find the unknown Quantity to be multiplied and divided by the same Quantity; now place down this Equation, rejecting only the Letter from that Quantity, where it multiplies and divides the unknown Quantity, as in the last Article; then transpose the Quantities as before, but if there are none to be transposed the Question is solved.

If any Quantities are connected with the unknown one by the Signs  $+$  or  $-$ , it will be most convenient for the Learner to transpose them before he begins to divide by the Rule just given.

Question 13. *A Person required another to tell him how many Shillings he had, by saying that if their Number was multiplied by 13, and if from that Product was subtracted 25, he should then have 170 Shillings. How many Shillings had he?*

Let  $a$  = the Number of Shillings the Person had,  $b = 13$ ,  $d = 25$ ,  $m = 170$ .

A Person had a certain Number of Shillings, which call	1	$a$
Which multiplied by 13, or $b$ , we have by Art. 9.	2	$ba$
From the Product, or $ba$ , if we subtract 25, or $d$ , we have	3	$ba - d$
Which Remainder $ba - d$ is by the Question to be equal to 170, or $m$ , hence	4	$ba - d = m$
Because $d$ is on the same Side of the Equation with the unknown Quantity, and connected by the Sign $-$ , therefore transpose $d$ , then	5	$ba = m + d$
There being no more Quantities to be transposed, and the unknown Quantity being multiplied by $b$ , therefore divide both Sides of the Equation by $b$ . Now $ba$ divided by $b$ gives $\frac{ba}{b}$ by Art. 27. and $m + d$ divided by $b$ , gives $\frac{m + d}{b}$ by Art. 28. therefore we have	6	$\frac{ba}{b} = \frac{m + d}{b}$

Because

Because  $b$  is in both Dividend and  
 Divisor of the Quantity  $\frac{ba}{b}$ , reject  $b$   
 by Art. 20. and putting down the  
 other Quantities without any Altera-  
 tion as in the foregoing Question, we  
 have

$$\left. \begin{array}{l} \text{Divisor of the Quantity } \frac{ba}{b}, \text{ reject } b \\ \text{by Art. 20. and putting down the} \\ \text{other Quantities without any Altera-} \\ \text{tion as in the foregoing Question, we} \\ \text{have} \end{array} \right\} 7 \left| a = \frac{m+d}{b}$$

From hence it appears that  $a$ , the unknown Quantity, is equal to the Sum of the two Numbers represented by  $m$  and  $d$ , divided by the Number represented by  $b$ .

The Number represented by $m$ is	-	-	170
The Number represented by $d$ is	-	-	25
The Sum is $m + d$ , or	-	-	195

And dividing 195, or  $m + d$ , by 13, or  $b$ , the Quotient is  $\frac{m+d}{b}$ , or 15, which is  $a$ , or the Number sought.

The Truth of which is thus proved from the Conditions of the Question.

I say the Person had	-	-	15 Shillings
For if that is multiplied by	=	-	13
			45
			15
The Product is	-	-	195
From which subtracting	-	-	25
There remains what the Question requires	-	-	170

*Question 14. A Butcher seeing a Drover going to Market with a Number of Sheep, asked how many there were; the Drover answered, if you multiply their Number by 9, and subtract 157 from that Product, and add 168 to the Remainder, I shall then have 2000 Sheep. How many Sheep had he?*

Let  $a$  = the Number of Sheep,  $b = 9$ ,  $d = 157$ ,  $m = 168$ ,  
 $p = 2000$ .

Then



The Drover had a certain Number	1	$a$
of Sheep, which call -		
Which multiplied by 9, or $b$ , we	2	$ba$
have by Art. 9. -		
From the Product subtracting 157, or	3	$ba - d$
$d$ , that is, connecting $d$ by the		
Sign —, we have -		
To which Remainder adding 168, or	4	$ba - d + m$
$m$ , we have by Art. 6. -		
This $ba - d + m$ is by the Question	5	$ba - d + m = p$
to be equal to 2000, or $p$ , hence		
we have -		
Now according to the Rule begin with	6	$ba - d = p - m$
transposing $m$ , and we have -		
Then transposing $d$ , we have -	7	$ba = p - m + d$
The Quantities being all transposed		
that were connected by the Signs +		
or —, and the unknown Quantity		
being multiplied by $b$ , therefore by		
the Rule divide both Sides of the		
Equation by $b$ , but $ba$ divided by $b$ ,	8	$\frac{ba}{b} = \frac{p - m + d}{b}$
gives $\frac{ba}{b}$ , and $p - m + d$ divided		
by $b$ , gives $\frac{p - m + d}{b}$ , by Art. 28.		
hence we have -		
Rejecting $b$ from the Quantity $\frac{ba}{b}$ ,		
because it is in both Dividend and	9	$a = \frac{p - m + d}{b}$
Divisor, and placing down the re-		
maining Parts of the Equation with-		
out any Alteration as before, we have		

The *Algebraic* Work is now finished, for the unknown Quantity  $a$  is on one Side of the Equation by itself, and it appears to be equal to the Number represented by  $p$ , subtracting from it the Number represented by  $m$ , adding to this Remainder the Number represented by  $d$ , and dividing this Sum by the Number represented by  $b$ .

The Number represented by $p$ is	-	-	2000
From which subtracting the Number represented by $m$ , which is	-	-	168
There remains $p - m$ , or	-	-	1832
To which adding the Number represented by $d$	-	-	157
The Sum is $p - m + d$ , or	-	-	1989

And dividing this 1989, or  $p - m + d$ , by 9, the Number represented by  $b$ , the Quotient is  $\frac{p - m + d}{b}$ , or 221, which is  $a$ , or the Number of Sheep the Drover had; and is proved by the Conditions of the Question thus.

I say the Number of Sheep were	-	-	221
For that being multiplied by	-	-	9
The Product is	-	-	1989
From which subtracting	-	-	157
There remains	-	-	1832
To which adding	-	-	168
The Sum is what the Question requires	-	-	2000

Question 15. *A Man being asked what he gave for his Horse, answered, if you multiply the Number of Pounds I gave by 5, and then add 15 to the Product, and from that Sum subtract 50, and to the Remainder adding 25, from which Sum subtracting 14, this Remainder will be equal to 81. What did he give for his Horse?*

Let  $a$  = what he gave for the Horse,  $b = 5$ ,  $d = 15$ ,  $c = 50$ ,  $p = 25$ ,  $m = 14$ ,  $x = 81$ .

Let the Pounds which the Person gave for the Horse be called	1	$a$
Which multiplied by 5, or $b$ , we have by Art. 9.	2	$ba$
To which Product if we add 15, or $d$ , we have by Art. 6.	3	$ba + d$
From this Sum subtracting 50, or $c$ , that is, connecting $c$ by the Sign —, we have	4	$ba + d - c$

To which Remainder adding 25,	5	$ba + d - c + p$
or $p$ , we have		
From which Sum subſtracting	6	$ba + d - c + p - m$
14, or $m$ , we have		
Which $ba + d - c + p - m$	7	$ba + d - c + p - m = x$
is by the Queſtion to be		
equal to 81, or $x$ , hence we		
have		
Now tranſpoſing $m$ , we have	8	$ba + d - c + p = x + m$
And tranſpoſing $p$ , we have	9	$ba + d - c = x + m - p$
And tranſpoſing $c$ , we have	10	$ba + d = x + m - p + c$
And tranſpoſing $d$ , we have	11	$ba = x + m - p + c - d$
The Quantities connected by		
the Signs $+$ or $-$ , being		
now all tranſpoſed, I obſerve		
the unknown Quantity to		
be multiplied by $b$ , there-		
fore divide every Term on		
both Sides of the Equa-		
tion by $b$ . Now dividing	12	$\frac{ba}{b} = \frac{x + m - p + c - d}{b}$
$ba$ by $b$ , it is $\frac{ba}{b}$ , and		
dividing $x + m - p + c - d$ by		
$b$ , we have $\frac{x + m - p + c - d}{b}$		
as in the foregoing Queſtions,		
hence we have		
Rejecting $b$ from the Quantity		
$\frac{ba}{b}$ , becauſe it is in both		
Dividend and Diviſor, and	13	$a = \frac{x + m - p + c - d}{b}$
placing down the reſt of the		
Equation without any Al-		
teration as before, and we		
have		

That is,  $a$ , the unknown Quantity, is equal to the Number represented by  $x$ , added to the Number represented by  $m$ , ſubſtracting from their Sum the Number represented by  $p$ , adding to this Remainder the Number represented by  $c$ , ſubſtracting from this Sum the Number represented by  $d$ , and dividing this Remainder by the Number represented by  $b$ .

Now $x$ is	-	-	-	-	81
To which adding $m$ , or	-	-	-	-	14
The Sum is $x + m$ , or	-	-	-	-	95
From which subtracting $p$ , or	-	-	-	-	25
There remains $x + m - p$ , or	-	-	-	-	70
To which adding $c$ , or	-	-	-	-	50
The Sum is $x + m - p + c$ , or	-	-	-	-	120
From which subtracting $d$ , or	-	-	-	-	15
There remains $x + m - p + c - d$ , or	-	-	-	-	105

Now dividing this 105, or  $x + m - p + c - d$ , by  $b$ , or 5,  
 the Quotient is  $\frac{x + m - p + c - d}{b}$ , or 21, which is equal to  $a$ ,  
 or Number of Pounds the Horfe cost.

Which is proved from the Conditions of the Question,  
 thus,

I say the Horfe cost	-	-	-	21 Pounds
For if that is multiplied by	-	-	-	5
The Product is	-	-	-	105
To which adding	-	-	-	15
The Sum is	-	-	-	120
From which subtracting	-	-	-	50
There remains	-	-	-	70
To which adding	-	-	-	25
The Sum is	-	-	-	95
From which subtracting	-	-	-	14
There remains what the Question requires	-	-	-	81

Question 16. *There is a certain Number which being multiplied by 7, if from this Product we subtract 21, and to the Remainder add 11, and from this Sum subtract 23, and add to the Remainder 33, this last Sum will be 210. What is the Number?*

Let  $a$  = the Number sought,  $b = 7$ ,  $d = 21$ ,  $x = 11$ ,  
 $c = 23$ ,  $p = 33$ ,  $r = 210$ .

Now

Now there is a certain Number sought, which call -	1	$a$
Which multiplied by 7, or $b$ , we have by Art. 9. -	2	$ba$
From which subtracting 21, or $d$ , that is, connecting $d$ by the Sign —, we have -	3	$ba - d$
To this adding 11, or $x$ , we have by Art. 6. -	4	$ba - d + x$
From which subtracting 23, or $c$ , that is, connecting $c$ by the Sign —, we have -	5	$ba - d + x - c$
To which adding 33, or $p$ , we have by Art. 6. -	6	$ba - d + x - c + p$
And this $ba - d + x - c + p$ is by the Question to be equal to 210, or $r$ , hence we have -	7	$ba - d + x - c + p = r$
The Question being now expressed in <i>Algebra</i> , begin the Solution by transposing $p$ , and then we have -	8	$ba - d + x - c = r - p$
Transposing $c$ we have -	9	$ba - d + x = r - p + c$
Transposing $x$ we have -	10	$ba - d = r - p + c - x$
Transposing $d$ we have -	11	$ba = r - p + c - x + d$
All the Quantities being now transposed that were connected by the Signs + or —, and the unknown Quantity being multiplied by $b$ , divide every Term, or both Sides of the Equation by $b$ , as in the last Example, and we have -	12	$\frac{ba}{b} = \frac{r - p + c - x + d}{b}$
Now reject $b$ out of the Quantity $\frac{ba}{b}$ , because it is in both Dividend and Divisor, and setting down the remaining Parts of the Equation, as in the last Question, and we have	13	$a = \frac{r - p + c - x + d}{b}$

To find what  $a$  is in Numbers.

The Number represented by $r$ , is	-	-	210
From which subtracting the Number represented by $p$ , which is	-	-	33
There remains $r - p$ , or	-	-	177
To which adding the Number represented by $c$	-	-	23
The Sum is $r - p + c$ , or	-	-	200
From which subtracting the Number represented by $x$	-	-	11
There remains $r - p + c - x$ , or	-	-	189
To which adding the Number represented by $d$	-	-	21
The Sum is $r - p + c - x + d$ , or	-	-	210

And dividing this 210 by  $b$ , or 7, the Quotient is  $\frac{r - p + c - x + d}{b}$ , or 30, which is equal to  $a$ , or the Number sought, and is thus proved.

I say the Number sought is	-	-	30
For if this is multiplied by	-	-	7
The Product is	-	-	210
From which subtracting	-	-	21
There remains	-	-	189
To which adding	-	-	11
The Sum is	-	-	200
From which subtracting	-	-	23
There remains	-	-	177
To which adding	-	-	33
The Sum is what the Question requires	-	-	210

Question 17. *A Gamester challenged another to play with him for as many Guineas as were in his Hand; but being asked how many they were, answered, if you multiply their Number by 10, and subtract 100 from the Product, and to the Remainder add 55, and from the Sum subtract 31, and adding to this Remainder 115, I shall then have 539 Guineas. How many had he?*

Let  $a$  = the Number of Guineas sought,  $b = 10$ ,  $c = 100$ ,  $d = 55$ ,  $m = 31$ ,  $x = 115$ ,  $p = 539$ .

Then

# To reduce an Equation, &c.

III

Then a Gamester had a certain		
Number of Guineas, which	1	$a$
call		
Which being multiplied by 10,	2	$ba$
or $b$ , we have by Art. 9.		
From which subtracting 100,	3	$ba - c$
or $c$ , we have		
To which adding 55, or $d$ , we	4	$ba - c + d$
have by Art. 6.		
From this subtracting 31, or	5	$ba - c + d - m$
$m$ , we have		
To which adding 115, or $x$ ,	6	$ba - c + d - m + x$
we have		
This by the Question is to be	7	$ba - c + d - m + x = p$
equal to 539, or $p$ , hence we		
have		
Then by transposing $x$ we have	8	$ba - c + d - m = p - x$
Transposing $m$ it is	9	$ba - c + d = p - x + m$
Transposing $d$ we have	10	$ba - c = p - x + m - d$
And transposing $c$	11	$ba = p - x + m - d + c$
Now divide by $b$ , as before di-	12	$\frac{ba}{b} = \frac{p - x + m - d + c}{b}$
rected, and we have		
And rejecting $b$ from $\frac{ba}{b}$ , and	13	$a = \frac{p - x + m - d + c}{b}$
placing down the rest as be-		
fore, then		

The Question being now solved in *Algebra*, we are to find what  $a$  is equal to in Numbers.

Now $p$ is equal to	539
From which subtracting $x$ , or	115
There remains $p - x$ , or	424
To this adding $m$ , or	31
The Sum is $p - x + m$ , or	455
From this subtracting $d$ , or	55
There remains $p - x + m - d$ , or	400
To this adding $c$ , or	100
The Sum is $p - x + m - d + c$ , or	500

But dividing this 500 by  $b$ , which is 10, the Quotient is 50, the Number of Guineas the Gamester had; and is thus proved from the Conditions of the Question.

I say the Gamester had	-	-	50 Guineas
For if that Number is multiplied by	-	-	10
The Product is	-	-	500
From which subtracting	-	-	100
There remains	-	-	400
To which adding	-	-	55
The Sum is	-	-	455
From which subtracting	-	-	31
There remains	-	-	424
To which adding	-	-	115
The Sum is what the Question requires	-	-	539

Question 18. *A Person being asked how many Hours it was past Noon, replied, if you multiply the Number of Hours past Noon by 7, and subtract 5 from the Product, and to the Remainder add 9, and from the Sum subtract 3, and to this Remainder adding 4, this Sum will be equal to 12. How many Hours was it past Noon, or what of the Clock was it?*

Let  $a$  = the Number of Hours it was past Noon, or the Number sought,  $m = 7$ ,  $p = 5$ ,  $d = 9$ ,  $c = 3$ ,  $b = 4$ ,  $x = 12$ .

Then there is a certain Number of Hours past Noon, which call	1	$a$
This multiplied by 7, or $m$ , we have by Art. 9.	2	$ma$
From which subtracting 5, or $p$ , we have	3	$ma - p$
To this adding 9, or $d$ , we have by Art. 6.	4	$ma - p + d$
From which subtracting 3, or $c$ , that is, connecting $c$ by the Sign —, we have	5	$ma - p + d - c$
To which adding 4, or $b$ , we have by Art. 6.	6	$ma - p + d - c + b$

Which



Which by the Question is to be } equal to 12, or $x$ , hence	7	$na - p + d - c + b = x$
Now transpose $b$ , and we have	8	$na - p + d - c = x - b$
Transposing $c$ , then	9	$ma - p + d = x - b + c$
Transposing $d$ , and	10	$na - p = x - b + c - d$
Transposing $p$ , we have	11	$ma = x - b + c - d + p$
Now dividing by $m$ , as in the } former Questions, and we } have	12	$\frac{ma}{m} = \frac{x - b + c - d + p}{m}$
Rejecting $m$ from the Quantity } $\frac{ma}{m}$ , as before, and we have }	13	$a = \frac{x - b + c - d + p}{m}$

The *Algebraic* Work being finished, we may find what  $a$  is in Numbers thus.

Now $x$ is equal to	-	-	-	12
From which subtracting $b$ , or	-	-	-	<u>4</u>
There remains $x - b$ , or	-	-	-	8
To which adding $c$ , or	-	-	-	<u>3</u>
The Sum is $x - b + c$ , or	-	-	-	11
From which subtracting $d$ , or	-	-	-	<u>9</u>
There remains $x - b + c - d$ , or	-	-	-	2
To which adding $p$ , or	-	-	-	<u>5</u>
The Sum is $x - b + c - d + p$ , or	-	-	-	7

And dividing this by  $m$ , or 7, the Quotient is 1, which is equal to  $a$ , or the Number of Hours it was past Noon, hence it was 1 of the Clock in the Afternoon.

Which is thus proved from the Conditions of the Question.

I say the Number of Hours past Noon were	-	-	-	1
For if that is multiplied by	-	-	-	<u>7</u>
The Product is	-	-	-	7
From which subtracting	-	-	-	<u>5</u>
There remains	-	-	-	2
To which adding	-	-	-	<u>9</u>
The Sum is	-	-	-	11
From which subtracting	-	-	-	<u>3</u>
There remains	-	-	-	8
To which adding	-	-	-	<u>4</u>
The Sum is what the Question requires	-	-	-	12

Q

T<sup>e</sup>

*To reduce an Equation by Involution.*

49. Hitherto there has been no Equation in which the unknown Quantity has had the radical Sign prefixt before it, or has been connected with known Quantities under the radical Sign; but as this is a Case which frequently happens, we are now to explain the Manner, how such Equations are managed.

If any Part of an Equation is a *surd* Quantity, but the unknown Quantity is not under the radical Sign, then there is no Occasion to clear this Equation of its *Surds*; but if the unknown Quantity is under the radical Sign, then the Equation must be cleared of its *Surds*.

And when there is a given Equation where the *unknown Quantity is under the radical Sign*, and there are known Quantities without the radical Sign on that Side of the Equation, and connected by the Signs  $+$  or  $-$ , transpose all those Quantities which are without the radical Sign, to the other Side of the Equation; then raise both Sides of the Equation to the Square, if the radical Sign expresses the Square Root, or to the Cube, if the radical Sign expresses the Cube Root, and so on; by which Means the Equation will be cleared of its *Surds*.

After this, if there are no known Quantities on the same Side of the Equation with the unknown one, the Question is solved; but if there are still known Quantities on the same Side of the Equation with the unknown Quantity, the Equation is to be reduced by some of the Methods before explained, at Art. 46, 47, 48.

The Square Root is expressed by this Sign  $\sqrt{\quad}$ , and the Cube Root by the same Sign with a 3 on the Top, thus  $\sqrt[3]{\quad}$ ; and if any Root is taken besides the Square Root, the Figure over the Sign shews what Root it is; but when it is only the Square Root, then there is generally no Figure over the Sign.

Question 19. *Two Gentlemen were talking of the Number of Acres there were in a Park, the Park-Keeper being present, and disposed to show his Learning; told them, that if they extracted the Square Root of the Number of Acres in the Park, from which square Root subtracting 5, this Remainder will be equal to 50. How many Acres were there in the Park?*

Let

Let  $a$  = the Number of Acres in the Park,  $b = 5$ ,  
 $d = 50$ .

Now there were a certain Number of Acres in the Park, which call	1	$a$
The Square Root of which, by Art. 33. is	2	$\sqrt{a}$
From which subtracting 5, or $b$ , that is, connecting $b$ by the Sign —, it is	3	$\sqrt{a} : - b$
Which $\sqrt{a} : - b$ by the Question is equal to 50, or $d$ , hence	4	$\sqrt{a} : - b = d$
The Question being now expressed in <i>Algebra</i> , and observing that $b$ is not under the radical Sign, therefore transpose $b$ , then	5	$\sqrt{a} = d + b$
Now all the Quantities being transposed, which were not under the radical Sign, square both Sides of the Equation, as the radical Sign expresses the Square Root. But the Square of $\sqrt{a}$ is $a$ , by Art. 43. and the Square of $d + b$ is $dd + 2db + bb$ , by Art. 32. and making these equal to one another, for the Square of equal Quantities or Numbers must be equal, and we have	6	$a = dd + 2db + bb$

Hence it appears that  $a$ , the unknown Quantity, is equal to the Square of the Number represented by  $d$ , added to twice the Product of the two Numbers represented by  $d$  and  $b$ , and this Sum added to the Square of the Number represented by  $b$ .

The Square of the Number represented by $d$ is $dd$ , or	2500
The Product of the two Numbers represented by $d$ and $b$ is $db$ , or 250, and twice that Product is	500
$2db$ , or	
The Sum is $dd + 2db$ , or	3000
The Square of the Number represented by $b$ is $bb$ , or	25
The Sum is $dd + 2db + bb$ , or 3025, which is	3025
$= a$ , or the Number sought	

Hence, I say, there were 3025 Acres in the Park, which is thus proved, from the Conditions of the Question.

The Number of Acres in the Park were	-	3025
Now the Square Root of that Number is	-	55
From which subtracting	-	5
There remains what the Question requires	-	50

Question 20. *A Person, who had been fortunate at Gaming, was asked how many Guineas he had won, to which he answered, that if the Square Root of their Number was extracted, from which Root subtracting 7, he should then have 16 Guineas. What Number of Guineas did he win?*

Let  $a$  = the Number of Guineas won,  $b = 7$ ,  $d = 16$ .

Now a Person won a Number of } Guineas, which call -	1	$a$
The square Root of which by Art. } 33. is -	2	$\sqrt{a}$
From which subtracting 7, or $b$ , we } have -	3	$\sqrt{a} - b$
Which $\sqrt{a} - b$ by the Question is } to be equal to 16, or $d$ , hence	4	$\sqrt{a} - b = d$
Now because $b$ is not under the } radical Sign, therefore transpose $b$ , } then -	5	$\sqrt{a} = d + b$
All the Quantities not under the ra- } dical Sign being now transposed, } in order to clear the Equation of } the Surd, raise both Sides of the } Equation to the Square or second } Power. But the Square of $\sqrt{a}$ is } $a$ , by Art. 43. and the Square of } $d + b$ is $dd + 2db + bb$ , by } Art. 32. and making these two } equal to one another, for the Square } of equal Quantities or Numbers } must be equal, and we have	6	$a = dd + 2db + bb$

That is to say, the unknown Quantity, or  $a$ , is equal to the Square of the Number represented by  $d$ , added to twice the Product of the two Numbers represented by  $d$  and  $b$ , to which Sum add the Square of the Number represented by  $b$ .

The

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The Square of the Number represented by $d$ is $dd$ , or	256
The Product of the two Numbers represented by $d$ and $b$ is $db$ , or 112, and twice that Product is $2db$ , or	224
The Sum is $dd + 2db$ , or	480
The Square of the Number represented by $b$ is $bb$ , or	49
The Sum is $dd + 2db + bb$ , or 529, which is equal to $a$ , or the Number sought	529

Therefore the Person won 529 Guineas; and is thus proved, from the Conditions of the Question.

I say the Number of Guineas he won was	529
For the Square Root of that Number is	23
And if from that Square Root we subtract	7
There remains what the Question requires	16

Question 21. *A Gentleman having sold his Estate, an impertinent illiterate Person asked him what he had sold it for, why, Sir, replied he, if you extract the Square Root of the Number of Guineas for which I sold it, and add 17 to that Number, this Sum will be equal to 317. How many Guineas had the Gentleman for his Estate?*

Let  $a$  = the Number of Guineas for which the Estate was sold,  $b = 17$ ,  $d = 317$ .

Now the Estate was sold for a	1	$a$
Number of Guineas, which call		
The square Root of which by Art. 33. is	2	$\sqrt{a}$
To which 17, or $b$ , being added, we have	3	$\sqrt{a + b}$
Which $\sqrt{a + b}$ by the Question is to be equal to 317, or $d$ , hence	4	$\sqrt{a + b} = d$
The Question being now expressed in Algebra, and because $b$ is not under the radical Sign, therefore transpose $b$ , and we have	5	$\sqrt{a} = d - b$
Now square both Sides of the Equation, and make them equal to one another, for the Reasons mentioned in the two last Questions, and we have	6	$a = dd - 2db + bb$

From

From hence we know that  $a$ , the unknown Quantity, is equal to the Square of the Number represented by  $d$ , subtracting from it twice the Product of the Numbers represented by  $d$  and  $b$ , and adding to the Remainder the Square of the Number represented by  $b$ .

The Square of the Number represented by $d$ is } $dd$ , or	100489
The Product of the two Numbers represented by } $b$ and $d$ is $db$ , or 5389, and twice that Product is } $2db$ , or	10778
Which subtracted, the Remainder is $dd - 2db$ , } or	89711
The Square of the Number represented by $b$ is } $bb$ , or	289
The Sum is $dd - 2db + bb$ , or 90000, which } is equal to $a$ , and is the Number sought	90000

And that the Estate was sold for 90000 Guineas, is thus proved from the Conditions of the Question.

I say the Estate was sold for	90000	Guineas
For the square Root of that is	300	
To which if we add	17	
The Sum is what the Question requires	317	

Question 22. *A young Gentleman, when he came of Age, asked his Guardian the annual Rent of the Estate his Father left him, to which he was answered, that if he extracted the square Root of the Number of Pounds for which the Estate was rented, and to this Root if he added 27, it would be equal to 100 Pounds. What was the annual Rent of the Estate?*

Let  $a$  = the Rent of the Estate,  $m = 27$ ,  $x = 100$ .

Now the Rent of the Estate was	1	$a$
The square Root of which by Art. 33. is	2	$\sqrt{a}$
To which 27, or $m$ , being added, we have	3	$\sqrt{a} + m$
Which by the Question is to be equal to 100, or $x$ , hence	4	$\sqrt{a} + m = x$

The

<p>The Question being now expressed in <i>Algebra</i>, begin by transposing <math>m</math>, for the Reasons mentioned in the former Questions, and then we have</p> <p>Now squaring both Sides of the Equation, to take away the radical Sign, as was done in the foregoing Questions, and then we have</p>	}	5	$\sqrt{a = x - m}$
	}	6	$a = xx - 2xm + mm$

And there being no more Quantities to be transposed, the Question is solved; for we may find the Value of  $a$  in Numbers from the *Algebraic* Work, thus:

The Square of the Number represented by $x$ is $xx$ , or	10000
The Product of the two Numbers represented by the Letters $x$ and $m$ is $xm$ , or 2700, and twice that Product is $2xm$ , or	5400
Which subtracted leaves $xx - 2xm$ , or	4600
The Square of the Number represented by $m$ is $mm$ , or	729
The Sum is 5329, or $xx - 2xm + mm$ , which is equal to $a$ , or the Number sought	5329

And that the annual Rent of the Estate was 5329 Pounds, is proved from the Conditions of the Question.

I say the annual Rent of the Estate was	5329 Pounds
For the square Root of that is	73
To which there being added	27
The Sum is what the Question requires	100

Question 23. To find that Number to which 1290 being added, if the square Root of this Sum is extracted, from which Root subtracting 29, the Remainder may be 71.

Let  $a$  = the Number sought,  $b = 1290$ ,  $d = 29$ ,  $x = 71$ .

<p>There is a Number sought, which I call</p> <p>To which 1290, or <math>b</math>, being added, we have</p> <p>The square Root of which Sum by Art. 34. is</p>	}	1	$a$
	}	2	$a + b$
	}	3	$\sqrt{a + b}$

From which subtracting 29, or }  
 $d$ , we have - - - }

Which by the Question is equal }  
 to 71, or  $x$ , hence - - }

Now begin the Solution with }  
 transposing  $d$ , it not being }  
 under the radical Sign, and }  
 then - - - }

All the Quantities on one Side }  
 of the Equation being now }  
 under the radical Sign, to }  
 take away that, as the un- }  
 known Quantity is under }  
 it, square both Sides of the }  
 Equation as before. Now the }  
 Square of  $\sqrt{a+b}$  is  $a+b$ , }  
 by Art. 43. and the Square }  
 of  $x+d$  is  $xx+2xd+dd$ , }  
 by Art. 32. and as the Squares }  
 of equal Numbers, or Quanti- }  
 ties, must be equal to one ano- }  
 ther, hence - - - }

Now transpose  $b$ , it being a }  
 known Quantity, and then }

$$4 \quad \sqrt{a+b} : -d$$

$$5 \quad \sqrt{a+b} : -d = x$$

$$6 \quad \sqrt{a+b} = x+d$$

$$7 \quad a+b = xx+2xd+dd$$

$$8 \quad a = xx+2xd+dd-b$$

From whence we may find the Value of  $a$  in Numbers.

The Square of the Number represented by  $x$  is  $xx$ , or 5041  
 The Product of the two Numbers represented by  $x$  }  
 and  $d$  is  $xd$ , or 2059, and twice that Product is } 4118  
 $2xd$ , or - - - - -

The Sum is  $xx+2xd$ , or - - - - 9159

The Square of the Number represented by  $d$  is  $dd$ , or 841

The Sum is  $xx+2xd+dd$ , or - - - 10000

From which subtracting the Number represented by  $b$  1290

There remains 8710, or  $xx+2xd+dd-b$ , }  
 which is  $=a$ , or the Number sought - - - 8710

And is thus proved from the Conditions of the Question.

I say



# To reduce an Equation, &c.

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I say the Number sought is	-	-	-	8710
For if to that we add	-	-	-	1290
The Sum is	-	-	-	10000
The square Root of which is	-	-	-	100
From which subtracting	-	-	-	29
There remains what the Question requires	-	-	-	71

Question 24. *A Person being asked his Age, replied, that if from my Age you subtract 11, and extract the square Root of the Remainder, to which Root adding 13, this Sum will be equal to 20. What was the Age of the Person?*

Let  $a$  = the Number of Years, or Age of the Person,  
 $b = 11$ ,  $m = 13$ ,  $d = 20$ .

Now the Age of the Person is  
 From which if we subtract 11,  
 or  $b$ , we have

The square Root of which by  
 Art. 34. is

To which adding 13, or  $m$ , we  
 have

Which by the Question is equal  
 to 20, or  $d$ ; hence we have

The Question being thus ex-  
 pressed in *Algebra*, and  $m$  not  
 being under the radical Sign,  
 therefore transpose  $m$ ; then

Now square both Sides of the  
 Equation, to clear it of the  
 Surd, as in the former Que-  
 stions. But the Square of  
 $\sqrt{a-b}$ , is  $a-b$ , by Art.  
 43. and the Square of  $d-m$   
 by Art. 32. is  $dd - 2dm + mm$ ;  
 then as the Squares of  
 equal Quantities are equal, we  
 have

And by transposing  $b$  we have

1	$a$
2	$a - b$
3	$\sqrt{a - b}$
4	$\sqrt{a - b} : + m$
5	$\sqrt{a - b} : + m = d$
6	$\sqrt{a - b} = d - m$
7	$a - b = dd - 2dm + mm$
8	$a = dd - 2dm + mm + b$

By which we find what  $a$  is in Numbers. Thus,

The Square of the Number represented by $d$ is $dd$ , or	400
The Product of the two Numbers represented by $d$ } and $m$ is $dm$ , or 260, and twice that Product is $2dm$ , or }	520
Which 520 subtracted from 400, leaves $dd - 2dm$ , } or $-120$ , (see the Numerical Work in Question 6.) }	$-120$
The Square of the Number represented by $m$ is $mm$ , or	169
Which 169 added to $-120$ , makes $dd - 2dm + mm$ , } or $+49$ , (see the Numerical Work in Question 6.) }	49
To which adding the Number represented by $b$	11
The Sum is 60, which I say is $= a$ , or the Age of the Person	60

And is proved from the Conditions of the Question, thus :

I say the Person was	60 Years old
For if from that you subtract	11
There remains	49
The square Root of which is	7
To which adding	13
The Sum is what the Question requires	20

### *To reduce an Equation by Evolution.*

50. This is done by the Extraction of Roots, for if after all the known Quantities have been carried to the other Side of the Equation from the unknown Quantity, it appears that one Side of the Equation is the *Square*, *Cube*, or any Power of the unknown Quantity, then extract such Root of both Sides of the Equation as will depress or lower this Power of the unknown Quantity to the *first* Power; that is, if one Side of the Equation is the *Square* of the unknown Quantity, then the *Square Root* must be extracted; and if it is the *Cube* of the unknown Quantity, then the *Cube Root* must be extracted, and so on, which depressing the unknown Quantity to the first Power, the Question is answered.

Question 25. *What is that Number, if to the Square of which there is 51 added, the Sum may be 100?*

Let  $a$  = the Number sought,  $b = 51$ ,  $m = 100$ .

Now

Now there is a Number sought, which	1	$a$
I call	2	$aa$
The Square of which by Art. 31. is	3	$aa + b$
To which 51, or $b$ , being added, we	4	$aa + b = m$
have	5	$aa = m - b$
And this $aa + b$ is by the Question to	6	$a = \sqrt{m - b}$
be equal to 100, or $m$ ; hence		
The Question being expressed in <i>Al-</i>		
<i>gebra</i> , begin and transpose $b$ ; then		
The known Quantities being now all		
on one Side of the Equation, and the		
other Side being $aa$ , or the Square		
of $a$ ; therefore by the Rule extract		
the square Root of both Sides of the		
Equation. Now the square Root of		
$aa$ is $a$ , by Art. 33. and the Square		
Root of $m - b$ is $\sqrt{m - b}$ , by Art.		
34. and as the square Root of equal		
Quantities must be equal, therefore		

Hence  $a$ , or the Number sought, is equal to the Number represented by  $m$ , subtracting from it the Number represented by  $b$ , and extracting the square Root of the Remainder.

The Number represented by $m$ , is	-	-	100
From which subtracting $b$ , or	-	-	51
There remains $m - b$ , or	-	-	49
The square Root of which is $\sqrt{m - b}$ , or 7, and			7
is equal to $a$ , or the Number sought	-	-	

And is thus proved :

I say the Number sought is	-	-	7
The Square of which is	-	-	49
To which adding	-	-	51
The Sum is what the Question requires	-	-	100

Question 26. *A Merchant had gained so many Pounds, that if from the Square of their Number is subtracted 101, and to the Remainder add 500, this Sum will be 3000 Pounds. What had the Merchant gained?*

Let  $a$  = the Gain of the Merchant,  $b = 101$ ,  $m = 500$ ,  
 $p = 3000$ .

Then a Merchant had gained a certain	}	1	$a$
Number of Pounds, called			
The Square of which is by Art. 31.	}	2	$aa$
From which subtracting 101, or $b$ , we			
have	}	3	$aa - b$
To which adding 500, or $m$ , we have			
This, by the Question, is to be equal	}	4	$aa - b + m$
to 3000, or $p$ ; hence			
By transposing $m$ we have	}	5	$aa - b + m = p$
By transposing $b$ it is			
By extracting the square Roots, as at the	}	6	$aa - b = p - m$
sixth Step of the last Example; then			
	}	7	$aa = p - m + b$
	}	8	$a = \sqrt{p - m + b}$

That is,  $a$  is equal to the Number represented by  $p$ , subtracting from it the Number represented by  $m$ , and adding to this Remainder the Number represented by  $b$ , and extracting the square Root of the Sum.

The Number represented by $p$ is	-	-	3000
From which subtracting $m$ , or	-	-	500
There remains $p - m$ , or	-	-	2500
To which adding $b$ , or	-	-	101
The Sum is $p - m + b$ , or	-	-	2601
The square Root of which is $\sqrt{p - m + b}$ , or			
51, equal to $a$ , the Number sought	-		51

And is thus proved, from the Conditions of the Question.

I say the Merchant gained	-	-	51 Pounds
For the Square of that is	-	-	2601
From which subtracting	-	-	101
There remains	-	-	2500
To which adding	-	-	500
The Sum is what the Question requires	-	-	3000

Question 27. *If to the Square of the Number of Miles a Person had travelled there is added 97, subtracting from the Sum 251, and adding to the Remainder 160, this Sum will be 10006. How many Miles had he travelled?*

Let

Let  $a$  = the Number of Miles he had travelled,  $b = 97$ ,  
 $m = 251$ ,  $x = 160$ ,  $z = 10006$ .

Then a Person had travelled a certain Number of Miles, called	}	1	$a$
The Square of which is by Article 31.		2	$aa$
To which adding 97, or $b$ , it is	}	3	$aa + b$
From which subtracting 251, or $m$ , gives		4	$aa + b - m$
To which adding 160, or $x$ , it is	}	5	$aa + b - m + x$
Which by the Question is to be equal to 10006, or $z$ , whence		6	$aa + b - m + x = z$
By transposing $x$ it is	}	7	$aa + b - m = z - x$
By transposing $m$ we have		8	$aa + b = z - x + m$
By transposing $b$ then	}	9	$aa = z - x + m - b$
By extracting the square Root, as at the eighth Step of the last Example, we have		10	$a = \sqrt{z - x + m - b}$

That is, from the Number represented by  $z$ , subtract the Number represented by  $x$ , to the Remainder add the Number represented by  $m$ , from which Sum subtract the Number represented by  $b$ , extract the square Root of the Remainder, and it will be the Number sought.

The Number represented by $z$ is	-	-	10006
From which subtracting $x$ , or	-	-	160
There remains $z - x$ , or	-	-	9846
To which adding $m$ , or	-	-	251
The Sum is $z - x + m$ , or	-	-	10097
From which subtracting $b$ , or	-	-	97
There remains $z - x + m - b$ , or	-	-	10000
The square Root of which, or $\sqrt{z - x + m - b}$ , is	}		
the Number sought	-	-	100

## PROOF.

I say the Person had travelled	-	-	100 Miles
For the Square of that is	-	-	10000
To which adding	-	-	97
The Sum is	-	-	10097
From which subſtracting	-	-	251
There remains	-	-	9846
To which adding	-	-	160
The Sum is what the Queſtion requires	-	-	10006

Queſtion 28. *A General, upon numbering his Army, found, that if from the Square of the Number of Men in his Army, there was ſubſtracted 3196, and to the Remainder adding 2721, from which Sum ſubſtracting 1711, there would remain 99997814. To find the Number of Men in the Army?*

Let  $a$  = the Number of Men in the Army,  $b = 3196$ ,  
 $m = 2721$ ,  $x = 1711$ ,  $z = 99997814$ .

The Number of Men in the Army } was	1	$a$
The Square of which is by Ar- ticle 31.	2	$aa$
From which ſubſtracting 3196, or $b$ , it is	3	$aa - b$
To which adding 2721, or $m$ , gives	4	$aa - b + m$
From which ſubſtracting 1711, or $x$ , we have	5	$aa - b + m - x$
Which by the Queſtion is equal to 99997814, or $z$ ; hence	6	$aa - b + m - x = z$
Fiſt, by tranſpoſing $x$	7	$aa - b + m = z + x$
By tranſpoſing $m$	8	$aa - b = z + x - m$
By tranſpoſing $b$	9	$aa = z + x - m + b$
By extracting the ſquare Root, as in the former Examples	10	$a = \sqrt{z + x - m + b}$

By Numbers thus :

$z$  is

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$z$ is in Numbers	-	-	-	99997814
To which adding $x$ , or	-	-	-	1711
The Sum is $z + x$ , or	-	-	-	99999525
From which subtracting $m$ , or	-	-	-	2721
There remains $z + x - m$ , or	-	-	-	99996804
To which adding $b$ , or	-	-	-	3196
The Sum is $z + x - m + b$ , or	-	-	-	100000000
The square Root of which is $a$ , or the Number fought	-	-	-	10000

Which is thus proved :

I say the Number of Men in the Army were	10000
For the Square of that is	100000000
From which subtracting	3196
There remains	99996804
To which adding	2721
The Sum is	99999525
From which subtracting	1711
There remains what the Question requires	99997814

51. These being the particular Methods by which Equations are reduced, or Questions answered, we shall now add some Examples where all these Methods are promiscuously used.

Question 29. *A Merchant broke for so many Pounds, that if their Number was multiplied by 4, and the Product divided by 6, and extracting the square Root of the Quotient, from which subtracting 60, there remains 40. What was the Sum for which the Merchant broke?*

Let  $a$  = the Number of Pounds fought,  $b = 4$ ,  $d = 6$ ,  $m = 60$ ,  $p = 40$ .

Then the Merchant broke for a	}	1	$a$	
Number of Pounds, called				
Which multiplied by 4, or $b$ , we				
have	-	}	2	$ba$
	-			
This divided by 6, or $d$ , we	}	3	$\frac{ba}{d}$	
have				-

The

The square Root of which } is by Art. 33. -	4	$\sqrt{\frac{ba}{d}}$
From this subtracting 60, } or $m$ , we have -	5	$\sqrt{\frac{ba}{d}} - m$
Which by the Question is } equal to 40, or $p$ , hence	6	$\sqrt{\frac{ba}{d}} - m = p$
Because $m$ is not under the } radical Sign, therefore	7	$\sqrt{\frac{ba}{d}} = p + m$
transpose it, by Art. 49. }		
Now squaring both Sides of } the Equation by Art. 49. }	8	$\frac{ba}{d} = pp + 2pm + mm$
And multiplying by $d$ , by } Art. 47. then -	9	$\frac{dba}{d} = dpp + 2dpm + dmm$
Rejecting $d$ from $\frac{dba}{d}$ , }		
and putting down the } other Quantities without	10	$ba = dpp + 2dpm + dmm$
any Alteration, as at }		
Art. 47. we have - }		
Dividing by $b$ , by Art. 48. }	11	$\frac{ba}{b} = \frac{dpp + 2dpm + dmm}{b}$
then - - - }		
Rejecting $b$ from $\frac{ba}{b}$ , and }		
putting down the other }	12	$a = \frac{dpp + 2dpm + dmm}{b}$
Quantities without any }		
Alteration, as at Art. }		
47, or 48, we have }		

In Numbers thus :

$$\begin{array}{r}
 dpp = 9600 \\
 + 2dpm = 28800 \\
 + dmm = 21600 \\
 \hline
 \text{Sum } 60000 \text{ or } dpp + 2dpm + dmm
 \end{array}$$

Now dividing 60000, or  $dpp + 2dpm + dmm$ , by 4, or  $b$ ,  
we have  $\frac{dpp + 2dpm + dmm}{b}$ , or 60000, divided by 4 = 15000,  
which is equal to  $a$ , or the Number of Pounds for which the  
Merchant broke.



P R O O F.

$$\begin{array}{r}
 15000 \\
 \underline{4} \\
 6)60000 \\
 \underline{10000} \quad (100 \text{ the square Root of } 10000 \\
 60 \\
 \underline{40} \text{ as the Question requires.}
 \end{array}$$

Question 30. *A Gentleman having bought a House, and being disposed to try the Knowledge of his Son in Algebra, told him, if the Number of Pounds the House cost was divided by 8, and that Quotient multiplied by 50, and extracting the square Root of the Product, to which adding 10, this Sum would be 60 Pounds. What did the House cost?*

Let  $a$  = the Price of the House,  $b = 8$ ,  $d = 50$ ,  $m = 10$ ,  $p = 60$ .

Now the Price of the } House is -	1	$a$
Which divided by 8, or $b$ , } it is -	2	$\frac{a}{b}$
This multiplied by 50, or } $d$ , we have -	3	$\frac{da}{b}$
The square Root of which } is, by Art. 33. -	4	$\sqrt{\frac{da}{b}}$
To which adding 10, or $m$	5	$\sqrt{\frac{da}{b}} + m$
This, by the Question, is } equal to 60, or $p$ , hence	6	$\sqrt{\frac{da}{b}} + m = p$
The Question being now } expressed in <i>Algebra</i> , and } $m$ not being under the } radical Sign, transpose it } by Art. 49. then -	7	$\sqrt{\frac{da}{b}} = p - m$
Now squaring both Sides of } the Equation, by Art. 49. }	8	$\frac{da}{b} = pp - 2pm + mm$
And multiplying by $b$ , by } Art. 47. - - - }	9	$b\frac{da}{b} = bpp - 2bpm + bmm$

Rejecting $b$ from $\frac{bda}{b}$ , and	}	10	$da = b p p - 2 b p m + b m m$
putting down the rest as			
at the twelfth Step of			
the last Question, then	}	11	$\frac{da}{d} = \frac{b p p - 2 b p m + b m m}{d}$
Dividing by $d$ , by Art. 48.			
then			
Rejecting $d$ from $\frac{da}{d}$ , and	}	12	$a = \frac{b p p - 2 b p m + b m m}{d}$
putting down the rest as			
at Art. 47, or 48. and			

In Numbers :

$$\begin{array}{r}
 b p p = 28800 \\
 - 2 b p m = \quad 9600 \\
 \hline
 19200 \\
 + b m m = \quad 800 \\
 \hline
 d = 5 \overline{) 20000}
 \end{array}$$

$400 = a$ , the Number of Pounds the House cost.

P R O O F.

$$\begin{array}{r}
 8 \overline{) 400} \\
 \underline{50} \\
 50 \\
 \underline{50} \\
 2500 \text{ (50 the Square Root of 2500)} \\
 \underline{10} \\
 60 \text{ as the Question requires.}
 \end{array}$$

C O N S E C T A R Y.

If the Reader compares the eighth, ninth, and tenth Steps of the last Work, he will find that to multiply any Fraction by its Denominator, or any Dividend by its Divisor, is only to reject the Denominator, or Divisor, from that Quantity, and multiply it into all the other Quantities; thus, the Equation at the eighth Step is  $\frac{da}{b} = p p - 2 p m + m m$ , which being multiplied by

its

its Denominator  $b$ , we have at the tenth Step  $da = b p p - 2 b p m + b m m$ ; the ninth Step, or  $\frac{b d a}{b} = b p p - 2 b p m + b m m$ , being only a more particular Illustration of the Work.

And by comparing the tenth, eleventh, and twelfth Steps of the same Work, it appears, that to divide any Quantity, by any Letter in that Quantity, is only to reject that Letter from the Quantity, and placing it as a Divisor to the other Quantities; thus, at the tenth Step, the Equation is  $da = b p p - 2 b p m + b m m$ , which being divided by  $d$ , gives at the twelfth Step  $a = \frac{b p p - 2 b p m + b m m}{d}$ ; the eleventh Step or  $\frac{da}{d} = \frac{b p p - 2 b p m + b m m}{d}$ , being only a more particular Illustration of the Work.

Therefore we shall for the future leave out such Steps as the ninth and eleventh: I did not choose to do it before, my Design being to make this curious Science as easy as possible.

*Question 31. A Running-Footman being sent of an Errand, was told, that if he squared the Number of Miles he was to run, and multiplied the Square by 4, and divided the Product by 40, to this Quotient adding 500, from which Sum subtracting 1400, and extracting the square Root of the Remainder, it would be 10. How many Miles was the Footman to run?*

Let  $a$  = the Number of Miles the Footman was to run,  
 $b = 4$ ,  $d = 40$ ,  $m = 500$ ,  $x = 1400$ ,  $p = 10$ .

The Number of Miles the Foot-	}	1		$a$
man was to run let be				
Which being squared is by	}	2		$a a$
Art. 31.				
This being multiplied by 4,	}	3		$b a a$
or $b$ , we have				
This being divided by 40, or $d$ ,	}	4		$\frac{b a a}{d}$
it is				
To which adding 500, or $m$ ,	}	5		$\frac{b a a}{d} + m$
gives				

From which subtracting 1400, } or $x$ , we have -	6	$\frac{b a a}{d} + m - x$
The square Root of which is } by Art. 34. -	7	$\sqrt{\frac{b a a}{d} + m - x}$
Which by the Question, is equal } to 10, or $p$ , therefore -	8	$\sqrt{\frac{b a a}{d} + m - x} = p$
Now square both Sides of the } Equation, by Art. 49. and	9	$\frac{b a a}{d} + m - x = p p$
By transposing $x$ , we have - }	10	$\frac{b a a}{d} + m = p p + x$
By transposing $m$ , we have	11	$\frac{b a a}{d} = p p + x - m$
By multiplying by $d$ by the } <i>Conseſſary</i> , Page 130. -	12	$b a a = d p p + d x - d m$
And dividing by $b$ by the <i>Con-</i> } <i>ſeſſary</i> , Page 130. -	13	$a a = \frac{d p p + d x - d m}{b}$
Now extracting the square Root, } by Art. 50. - -	14	$a = \sqrt{\frac{d p p + d x - d m}{b}}$

In Numbers :

$$\begin{array}{r}
 d p p = 4000 \\
 + d x = 56000 \\
 \hline
 60000 \\
 - d m = - 20000 \\
 \hline
 b = 4) 40000
 \end{array}$$

10000 (100 the square Root of 10000, hence  
the Footman was to run 100 Miles.

### P R O O F.

$$\sqrt{\frac{4 a a}{40} + 500 - 1400} = 10$$

I have not drawn out the Proof of the last Question into Particulars, but only expreſſed it at once; that is, four times the Square of  $a$  (which is found to be 100) being divided by 40, if to this Quotient we add 500, and from this Sum ſubſtract 1400, the ſquare Root of this Remainder will be equal to 10. And now I ſhall expreſs all the Conditions of the Question at the firſt,  
Equation,

Equation, that the Learner may form some little Judgment in what Manner to shorten his Work; and if he conceives how the Proof of the last Question is expressed, it will easily lead him to the Knowledge of expressing the Conditions of the Question, or raise such Equations as arise from the Question without particularizing every Circumstance. But if the Learner finds any Difficulty in this, he may proceed as before.

*Question 32. A Gentleman who had been at the Gaming-Tables, and losing, some of his Acquaintance laughing at him for his Folly, asked how much he had lost; to which he answered, if you square the Number of Pounds I have lost, and divide that by 4, multiplying this Quotient by 10, to which Product add 3900, then extracting the square Root of this Sum, from which subtracting 80, the Remainder will be equal to 90. How much had he lost?*

Let  $a$  = the Number of Pounds lost,  $b = 4$ ,  $d = 10$ ,  $m = 3900$ ,  $p = 80$ ,  $z = 90$ .

Then by the Question	1	$\sqrt{\frac{daa}{b} + m} : -p = z$
By transposing $p$ , it not being under the radical Sign, by Art. 49. we have -	2	$\sqrt{\frac{daa}{b} + m} = z + p$
By squaring both Sides of the Equation, by Art. 49. then -	3	$\frac{daa}{b} + m = zz + 2zp + pp$
By transposing $m$ , it is	4	$\frac{daa}{b} = zz + 2zp + pp - m$
Multiplying by $b$ by the Confectary, Page 130. -	5	$daa = bzz + 2bzp + bpp - bm$
Dividing by $d$ by the same - -	6	$aa = \frac{bzz + 2bzp + bpp - bm}{d}$
Extracting the square Root, by Art. 50. }	7	$a = \sqrt{\frac{bzz + 2bzp + bpp - bm}{d}}$

In Numbers :

$dzz$

$$\begin{array}{r}
 bz z = 32400 \\
 2bz p = 57600 \\
 b p p = 25600 \\
 \hline
 115600 \\
 -bm = -15600 \\
 \hline
 d = 1|0 \overline{) 10000|0} \\
 \hline
 10000 (100 = a, \text{ the Number of Pounds left.} \\
 \hline
 1 \\
 \hline
 0
 \end{array}$$

P R O O F.

$$\sqrt{\frac{10aa}{4}} + 3900 : -80 = 90$$

*To reduce an Equation when the unknown Quantity is in several Terms.*

52. When the unknown Quantity is in more Terms than one, bring all those Terms which have the unknown Quantity to one Side of the Equation, taking Care that the *greatest Co-efficient* of the unknown Quantity has at last the *affirmative* Sign, and carrying all the Quantities that are known on the other Side of the Equation; then divide both Sides of the Equation by *all the Co-efficients* of the unknown Quantity, connected with the same Signs of  $+$  and  $-$ , as they then happen to have, which will reduce the Equation, as in the following Examples.

If the unknown Quantity should be in more than two Terms, transpose those Terms in such a Manner, that the Sum of the *positive* Co-efficients of the unknown Quantity may exceed the Sum of the *negative* Co-efficients of the unknown Quantity, and then divide as before directed.

Question 33. *There is a certain Number which being multiplied by 10, if this Product is divided by 2, to this Quotient adding 19, and subtracting 99 from that Sum, the Remainder will be equal to the Number sought.*

Let  $a$  = the Number sought,  $b = 10$ ,  $d = 2$ ,  $m = 19$ ,  $z = 99$ .

By the Question	-	-	1	$\frac{ba}{d} + m - z = a$
By transposing $z$	-	-	2	$\frac{ba}{d} + m = a + z$
By transposing $m$	-	-	3	$\frac{ba}{d} = a + z - m$
By multiplying by $d$ by the Confectary, Page 130.	-	-	4	$ba = da + dz - dm$
Because $d$ is less than $b$ , trans- pose $da$ , that both the Terms which have the unknown Quantity, may be on the same Side of the Equation, then	-	-	5	$ba - da = dz - dm$
And dividing according to the Rule by $b - d$ , the two Co- efficients of $a$ , we have	-	-	6	$a = \frac{dz - dm}{b - d} = 20$ the Number sought.

P R O O F.

$$\frac{10a}{2} + 19 - 99 = a$$

The Division at the fifth and sixth Steps, *viz.* that  $ba - da$ , divided by  $b - d$ , should leave only  $a$ , may perhaps a little perplex the Learner; and if it does, I advise him to examine *Art.* 10. where he may observe, that in multiplying any compound Quantity by a single Letter, that Letter goes into every Term of the Product, therefore the Multiplier is not so many Times that Letter as the Number of Terms are in which that Letter is found, but only that single Letter multiplied successively into all the other Quantities; hence, if this Product is to be divided by all those Quantities, the Quotient will be the single Letter, and not so many Times that Letter as the Number of Terms are in which it is found. See farther the Proof of the Question 38. and *Art.* 22.

Question 34. *A Gentleman bought an Estate for so many Pounds, that if they were multiplied by 4, and this Product divided by 5, from which Quotient subtracting 600, and adding to the Remainder 6 Times what the Estate cost, this Sum will be equal to 6200 Pounds. How much did the Estate cost?*

Let

Let  $a$  = the Number of Pounds the Estate cost,  $b = 4$ ,  
 $d = 5$ ,  $m = 600$ ,  $p = 6$ ,  $x = 6200$ .

Then by the Question -

$$1 \quad \frac{ba}{d} - m + pa = x$$

By transposing  $m$ , we have

$$2 \quad \frac{ba}{d} + pa = x + m$$

Multiplying by  $d$  by the Con-  
*jectary*, Page 130 -

$$3 \quad ba + dpa = dx + dm$$

Dividing by  $b + dp$ , the Co-  
 efficient of  $a$ , as in the last  
 Question, and we have -

$$4 \quad a = \frac{dx + dm}{b + dp} = 1000$$

Therefore the Estate cost 1000 Pounds.

### PROOF.

$$\frac{4a}{5} - 600 + 6a = 6200$$

*Question 35. A Person had a certain Number of Shillings, which multiplied by 4, this Product being divided by 11, to the Quotient adding 90, and from this Sum taking away 30, the square Root of this Remainder will be equal to the square Root of the Number of Shillings sought, after being diminished by 10.*

Let  $a$  = the Number of Shillings sought,  $b = 4$ ,  $d = 11$ ,  
 $x = 90$ ,  $p = 30$ ,  $z = 10$ .

Then by the Question -

$$1 \quad \sqrt{\frac{ba}{d} + x - p} = \sqrt{a - z}$$

Because there is no Quantity on  
 each Side of the Equation but  
 what is under the radical Sign,  
 therefore square both Sides of  
 the Equation, by Art. 49.

$$2 \quad \frac{ba}{d} + x - p = a - z$$

Multiplying by  $d$  by the Con-  
*jectary*, Page 130. -

$$3 \quad ba + dx - dp = da - dz$$

Because  $d$ , one Co-efficient of  $a$ ,  
 is greater than  $b$ , the other  
 Co-efficient of  $a$ , transpose  
 $ba$ , then -

$$4 \quad dx - dp = da - dz - ba$$

Transposing



# To reduce an Equation, &c. 137

Transposing $dz$	-	-	5	$  dz + dx - dp = da - ba$
Dividing by $d - b$ , the two	}	Co-efficients of $a$ , as at Question 33. Step 6. we have	6	$  a = \frac{dz + dx - dp}{d - b} = 110$
				(the Number sought.)

If the Learner chooses to have the unknown Quantity on the left Side of the Equation, he might have put the 5th Step thus,  $da - ba = dz + dx - dp$ , this being only to change the Sides of the Equation, not to alter their Value.

## P R O O F.

If  $a = 110$ , then  $\sqrt{\frac{4a}{11} + 90 - 30} = \sqrt{a - 10}$

Question 36. *A Running-Footman forward to show his Learning, being in Company, said, if the Number of Miles he had run was multiplied by 7, to which Product adding 550, and subtracting 20 from that Sum, and dividing the Remainder by 10, the square Root of the Quotient will be the same, as if you added 14 Miles to those he had run, and extracted the square Root of that Sum.*

Let  $a =$  the Number of Miles he had run,  $b = 7$ ,  $d = 550$ ,  $m = 20$ ,  $p = 10$ ,  $x = 14$ .

Then by the Question	-	1	$  \sqrt{\frac{ba + d - m}{p}} = \sqrt{a + x}$
There being no Quantity without the radical Sign, therefore square both Sides of the Equation as at the second Step of the last Question	}	2	$  \frac{ba + d - m}{p} = a + x$
Multiplying by $p$ by the Consequence, Page 130.		3	$  ba + d - m = pa + px$
Because $p$ , one Co-efficient of $a$ , is greater than $b$ , the other Co-efficient of $a$ , therefore transpose $ba$	}	4	$  d - m = pa + px - ba$
By transposing $px$		5	$  d - m - px = pa - ba$
Dividing by $p - b$ , the two Co-efficients of $a$ , as at Question 33. Step. 6. we have	}	6	$  a = \frac{d - m - px}{p - b} = 130$ the
		(Number of Miles required,	

## P R O O F.

$$\sqrt{\frac{7a + 550 - 20}{10}} = \sqrt{a + 14}$$

To reduce an Equation when the same Quantity, either known or unknown, is in every Term of the Equation.

53. In any Algebraic Operation, if the same Quantity, either known or unknown, is in every Term of any Equation, then divide every Term of the Equation by that Quantity, which will reduce the Equation to more simple Terms, as in the following Questions.

Question 37. To find a Number which multiplied by 4, and the Product added to the same Number multiplied by 56, and divided by 7, this Sum will be equal to the Square of the Number sought.

Let  $a$  = the Number sought,  $b = 4$ ,  $d = 56$ ,  $m = 7$ .

Then by the Question	-	}	1	$ba + \frac{da}{m} = aa$
Multiplying by $m$ by the		}	2	$m ba + da = m aa$
<i>Confessary</i> , Page 130.		}		
Because $a$ is in every Term		}	3	$mb + d = ma$
of the Equation, divide		}		
by $a$ , then		}		
Dividing by $m$ , the Co-		}	4	$a = \frac{mb + d}{m} = 12$ the Num-
efficient of $a$ , by the <i>Con-</i>		}		ber sought.
<i>fectary</i> , Page 130. and		}		

## P R O O F.

$$4a + \frac{56a}{7} = aa$$

Question 38. There are two Towns at such a Distance, that if the Number of Miles between them is multiplied by 79, and this Product added to their Distance, the square Root of this Sum will be equal to the Distance of the two Towns multiplied by 2.

Let

Let  $a$  = the Distance of the Towns,  $b = 79$ ,  $m = 2$ .

Then by the Question	-	-	1	$\sqrt{ba + a} = ma$
There being no rational Quantities on	}	the same Side of the Equation where	2	$ba + a = mma$
the radical Sign is, square both Sides				
of the Equation, and	}	Dividing by $a$ , it being in every Term	3	$b + 1 = mma$
of the Equation, and				
Dividing by $m$ , the Co-efficient of $a$	}		4	$a = \frac{b + 1}{mm} = 20$

Hence the Distance between the two Towns is 20 Miles.

### PROOF.

$$\sqrt{79a + a} = 2a$$

If the Reader does not easily conceive that dividing  $ba + a$ , or  $ba + 1a$  at the second Step, by  $a$ , gives  $b + 1$ , as at the third Step, I advise him to consider what is said at Question 33; to which may be added, that  $b + 1 \times a = ba + a$ , whereas  $b + 1 \times 2a = 2ba + 2a$ , a Product very different from  $ba + a$ . Or it may be explained thus,  $\frac{ba + 1a}{a} = b + 1$ , the  $a$  being rejected by Art. 22 and 26.

### *The Manner of registering the Steps of an Algebraic Operation explained.*

54. Having explained to the young Analyst the different Methods of managing Equations, to save the Trouble of using so many Words, I shall now show him the Method of *registering the Steps*, introduced by the ingenious Dr. John Pell.

To *register the Steps* of an Analytic Operation is only to express in the Margent of the Work by *Symbols*, instead of Words, what has been done; and to render it as easy as may be to the Learner, we shall resume the Work of one of the former Questions, and express by Words what is done in one Column, in another Column express the same Thing by *Symbols*, or Characters, and in the third Column place the Work itself, that by comparing the Operation with the Observations that follow it, the Reader may the more easily understand the Manner of *registering the Steps*.

**Question 39.** *A Running-Footman being sent of an Errand, was told, that if he squared the Number of Miles he was to run, and multiplied it by 4, and divided the Product by 40, to this Quotient adding 500, from which Sum subtracting 1400, and extracting the Square Root of the Remainder, it would be 10. How many Miles was the Footman to run? (this is Quest. 31.)*

Let  $a$  = the Number of Miles the Footman was to run,  
 $b = 4$ ,  $d = 40$ ,  $m = 500$ ,  $x = 1400$ ,  $p = 10$ .

Then, by the Question, we have }  
 the same Equation as at the }  
 eighth Step, Question 31. }

		1	$\sqrt{\frac{b a a}{d} + m - x} = p$
Squaring both Sides of the Equation, or involving them to the second Power, by Art. 49.	1 $\odot$ 2	2	$\frac{b a a}{d} + m - x = p p$
By transposing $x$ at the second Equation	2 $+$ $x$	3	$\frac{b a a}{d} + m = p p + x$
By transposing $m$ at the third Equation	3 $- m$	4	$\frac{b a a}{d} = p p + x - m$
Multiplying the fourth Equation by $d$	4 $\times d$	5	$b a a = d p p + d x - d m$
Dividing the fifth Equation by $b$	5 $\div b$	6	$a a = \frac{d p p + d x - d m}{b}$
Extracting the square Root of the sixth Equation, by Art. 50.	6 $\cup$ 2	7	$a = \sqrt{\frac{d p p + d x - d m}{b}} = 10$

For another Instance let us take Question 33.

**Question 40.** *There is a certain Number, which being multiplied by 10, if this Product is divided by 2, to this Quotient adding 19, and subtracting 99 from that Sum, the Remainder will be equal to the Number sought.*

Let  $a$  = the Number sought,  $b = 10$ ,  $d = 2$ ,  $m = 19$ ,  
 $x = 99$ .

Then

Then by the Question -	-	1	$\frac{ba}{d} + m - z = a$
By transposing $z$ from the first Equation -	$1 + z$	2	$\frac{ba}{d} + m = a + z$
By transposing $m$ from the second Equation -	$2 - m$	3	$\frac{ba}{d} = a + z - m$
Multiplying the third Equation by $d$ -	$3 \times d$	4	$ba = da + dz - dm$
By transposing $da$ from the fourth Equation	$4 - da$	5	$ba - da = dz - dm$
Dividing the fifth Equation by $b - d$ , the two Co-efficients of $a$ , by Art. 52.	$5 \div b - d$	6	$a = \frac{dz - dm}{b - d} = 20$

From these two Examples we may observe, that to *register* any Operation, is only to put down the Figure which stands in the Column against that Equation, from which we intend to raise the next Equation, and after that the *Sign* of either *Addition*, *Subtraction*, *Multiplication*, *Division*, *Involution* and *Evolution*, according as the Case requires, and after this the Quantity which suffers the Alteration.

Thus at Question 39, the first Equation being raised or involved to the second Power produces the second Equation, therefore, I say in the *Register* 1 & 2, that is, the first Equation involved to the second Power gives the second Equation, and in the same Operation.

Because the fourth Equation is produced from the third, by transposing  $m$  with the Sign  $-$ , therefore in the *Register* I say 3  $- m$ , that is, the third Equation  $- m$ , produces the fourth Equation. And,

As the fifth Equation is produced from the fourth by multiplying by  $d$ , therefore I say in the *Register* 4  $\times d$ , that is, the fourth Equation multiplied by  $d$ , produces the fifth Equation. And,

As the sixth Equation is produced from the fifth by dividing by  $b$ , therefore, I say in the *Register* 5  $\div b$ , that is, the fifth Equation divided by  $b$ , produces the sixth Equation. And,

As the seventh Equation is produced from the sixth by extracting the square Root, I say in the *Register* 6  $\sqrt{\phantom{x}}$  2, that is, the sixth Equation having the Square Root extracted, produces the seventh Equation.

Whence, as I said above, to register any Operation, is only to put down whether it is the first, second, third, fourth, or any other Equation, which suffers the Alteration, and from which the new Equation is raised; and after that Figure to express in Characters, or Signs, the Alteration that is then made to gain the new Equation.

---

*The Method of resolving Questions that contain two Equations, and two unknown Quantities.*

55. **T**HE foregoing Questions requiring only one unknown Number to be found, their Conditions were all expressed in one Equation, which Equation being reduced by the Rules already delivered, the Question was answered.

But if the Question requires two unknown Quantities to be found, then there are generally raised two Equations from the Question, each of them including both the unknown Quantities; which may be resolved by this

R U L E.

Find what the same unknown Quantity is *equal* to in each of the two Equations, which arise from the Conditions of the Question, then make these two Equations *equal* to one another, and in this Equation there will be but *one unknown Quantity*, consequently if this Equation is reduced by the Rules already given at Art. 46 to 53. we shall find what this unknown Quantity is.

And to find the Value of an unknown Quantity in any Equation, is only to find what it is equal to, therefore all the other Quantities, whether known or unknown, must be carried to the other Side of the Equation by the Directions at Art. 46 to 53. and then it will appear to what this unknown Quantity is equal, as this makes one Side of the Equation, the other Side of the Equation being known Quantities, with the other unknown Number or Quantity sought.

Finding

## The Method of resolving Questions, &c. 143

Finding the Value of the same unknown Quantity in each of the given Equations, and making these two Equations equal to one another, which clears the Work of that unknown Quantity whose Value was found, is called *exterminating an unknown Quantity*.

Question 41. To find two Numbers, if the greater is added to the lesser, the Sum is 262.

But if from the greater you subtract the lesser, the Remainder is 144.

Let  $a$  = the greater Number, and  $e$  = the lesser Number sought,  $b = 262$ ,  $x = 144$ .

Now the Sum of the two Numbers in Algebra is $a + e$ , which is equal to 262, or $b$ , hence we have	1	$a + e = b$	}	By the Question
And the lesser Number being subtracted from the greater is $a - e$ , which is equal to 144, or $x$ , hence we have	2	$a - e = x$		

The Conditions of the Question being now expressed, there appears in the above two Equations, two unknown Quantities  $a$  and  $e$ , therefore according to the Rule find what  $a$  is equal to in the first Equation, by transposing  $e$ .

Now find what $a$ is equal to in the second Equation, by transposing $e$	3	$a = b - e$
	4	$a = x + e$

Therefore make the third and fourth Equations equal to one another, for they are both equal to the same Quantity  $a$ , which exterminates that unknown Quantity: this Step is *registered* by placing the 3 and 4 with a Point between them as in the Work, which expresses that the fifth Equation is from comparing the third and fourth Equation together

$$3 \cdot 4 \mid 5 \mid x + e = b - e$$

The

The unknown Quantity  $e$  being on both Sides of the Equation, bring it on one Side of the Equation, by Art. 52.

$$\begin{array}{r|l} 5 + e & 6 \\ 6 - x & 7 \\ \hline 7 \div 2 & 8 \end{array} \quad \begin{array}{l} x + 2e = b \\ 2e = b - x \\ e = \frac{b - x}{2} \end{array}$$

Here it appears that  $e$ , or the lesser Number sought, is equal to  $b$ , or 262, subtracting from it  $x$ , or 144, and dividing the Remainder by 2.

When any Equation is divided by an absolute Number, as the seventh Equation is divided by 2, place them in the *Register* as usual, but draw a Line over the 2 to distinguish that it is an absolute Number by which you divide, and not by the second Equation in the Work.

$$\begin{array}{r} \text{Now } b = 262 \\ - x = - 144 \\ \hline 2) 118 \end{array}$$

$59 = e$ , the lesser of the two Numbers sought.

It being now known what  $e$  is in Numbers, we may find  $a$  by the third or fourth Equation, and by the third Equation we have  $a = b - e$ .

$$\begin{array}{r} \text{But } b = 262 \\ - e = - 59 \\ \hline \end{array}$$

$203 = a$ , the greater of the two Numbers sought.

Whence 203 and 59 are the two Numbers required in the Question, and is thus proved from their Conditions.

The greater Number is	203	-	-	203
The lesser Number is	59	-	-	59
	262	Sum		144
				Remains.

Question 42. Two Men discoursing of their Money, found that if the Number of Shillings each had were added together, the Sum would be 38.

But if from him that had the greater Number of Shillings, there be subtracted twice the Number of Shillings the other Person had, there would remain 5. How many had each Man?

Let



Let  $a$  = the greater Number of Shillings,  $e$  = the lesser Number of Shillings,  $b = 38$ ,  $x = 5$ .

And because the Sum of their Shillings, or $a + e$ , was 38, or $b$ , hence	$\left. \begin{array}{l} 1 \quad a + e = b \\ 2 \quad b - 2e = x \\ \hline 3 \quad a = b - e \\ 4 \quad a = x + 2e \end{array} \right\} \text{By the Question}$
And twice the lesser Number being taken from the greater, or $a - 2e$ , was equal to 5, or $x$ , hence	
Now to find the Value of $a$ in the first Equation, transpose $e$ .	
And to find the Value of $a$ in the second Equation, transpose $2e$ .	

Make the third and fourth Equations equal to one another, because they are both equal to the same Quantity  $a$ , and register it as directed in the last Question; and this exterminates the unknown Quantity  $a$ .

$$3 \cdot 4 \mid 5 \mid x + 2e = b - e$$

The unknown Quantity  $e$  being on both Sides of the Equation, bring it on one Side of the Equation, by Art. 52.

$$\begin{array}{l|l} 5 + e & 6 \mid x + 3e = b \\ 6 - x & 7 \mid 3e = b - x \\ 7 \div 3 & 8 \mid e = \frac{b - x}{3} \end{array}$$

Hence the Question is answered, for  $b = 38$

$$\begin{array}{r} -x = -5 \\ \hline 3 \mid 33 \end{array}$$

$11 = e$ , the lesser Number of Shillings.

And as  $e$  is now known, we may find what  $a$  is by the third or fourth Equation; taking the fourth Equation, we have

$$\begin{array}{r}
 x = 5 \\
 2e = 22 \\
 \hline
 27 = a, \text{ the greater Number of Shillings.}
 \end{array}$$

## P R O O F.

The greater Num- ber of Shillings	}	27	-	-	-	27
The lesser Num- ber of Shillings	}	11				
	}	Sum 38				
					Twice the lesser Number of Shillings	}
						22
						Remains 5

**Question 43.** *Two Men laying a Wager concerning the Number of Sheep in two Drove, as they could not decide it, appealed to a third Person, who told them, that if 31 was added to the Number of Sheep in the greatest Drove, that Sum would be equal to twice the Number of Sheep in the least Drove.*

*But if they added 44 to the Number of Sheep in the least Drove, that Sum would be as many as were in the greatest Drove, and desired they would now find the Number of Sheep in each Drove.*

Let  $a$  = the Number of Sheep in the greatest Drove,  $e$  = the Number of Sheep in the least Drove,  $x = 31$ ,  $d = 44$ .

Now the Number of Sheep in the greatest Drove being added to 31, is equal to twice the Number of Sheep in the lesser Drove, hence	1	$a + x = 2e$	By the Question
And the Number of Sheep in the least Drove when added to 44, being equal to the Number of Sheep in the greatest Drove, we have	2	$e + d = a$	
	3	$a = 2e - x$	

Now by the third Equation,  $a$  is equal to  $2e - x$ , and by the second Equation,  $a$  is equal to  $e + d$ , therefore make these Equations equal to one another, for they are both equal to the same Quantity  $a$ , which exterminates  $a$ , as before.

$$\begin{array}{l|l|l} 2 \cdot 3 & 4 & 2e - x = e + d \\ 4 - e & 5 & e - x = d \\ 5 + x & 6 & e = d + x \end{array}$$

$$d = 44$$

$$x = 31$$

75 = e, the Number of Sheep in the least Drove.

Then having found e, we may find a by the second Equation.

$$e = 75$$

$$d = 44$$

119 = a, the Number of Sheep in the greatest Drove.

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$$119$$

$$31$$

150 which is twice the Number of Sheep in the least Drove.

$$75$$

$$44$$

119 which is the Number of Sheep in the greatest Drove.

Question 44. Two Gentlemen who had sold their Estates, by comparing what each Estate was sold for, found, that twice the Sum of what both the Estates was sold for was 11468 Pounds:

And if what the least Estate was sold for, be subtracted from what the greatest Estate was sold for, there will remain 1408 Pounds. For how much was each Estate sold?

Let a = the Number of Pounds the greatest Estate was sold for, e = the Number of Pounds the least Estate was sold for, b = 11468, x = 1408.

$$\begin{array}{l|l|l} \text{By the first Condition} & - & 1 \\ \text{By the second Condition} & - & 2 \\ \hline \text{Find the Value of } a, \text{ in the} & & \\ \text{first Equation.} & & \\ 1 - 2e & 3 & 2a = b - 2e \\ 3 \div 2 & 4 & a = \frac{b - 2e}{2} \end{array}$$

U 2

Now

Now find the Value of  $a$ , in the second Equation,

$$2 + e \mid 5 \mid a = x + e$$

Make the fourth and fifth Equations equal to one another, because they are both equal to the same Quantity  $a$ , and therefore must be equal to one another, by which  $a$  will be exterminated.

$4 \cdot 5$	$6$	$x + e = \frac{b - 2e}{2}$	* Here we have only to find what $e$ is by the Rules already delivered, at Art. 46 to 53.
$6 \times 2$	$7$	$2x + 2e = b - 2e$	
$7 + 2e$	$8$	$2x + 4e = b$	
$8 - 2x$	$9$	$4e = b - 2x$	
$9 \div 4$	$10$	$e = \frac{b - 2x}{4} = 2163$ , the Pounds for	

which the least Estate was sold; and  $e$  being now known, then

By the fifth Step  $11 \mid a = x + e = 3571$ , the Pounds for which the greatest Estate was sold.

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Now if  $a = 3571$ , and  $e = 2163$ , then  $2a + 2e = 11468$ , and  $a - e = 1408$ .

*Question 45. Two Gamesters, A and B, found, that if twice the Number of Pounds won by A was added to what had been won by B, the Sum was 48 Pounds:*

*And if what had been won by A was added to three times what had been won by B, the Sum was 39 Pounds. What was the Sum won by each Gamester?*

Let  $a$  = the Pounds won by A,  $e$  = the Pounds won by B,  $b = 48$ ,  $x = 39$ .

By the first Condi-	}	$1$	$2a + e = b$
tion	-		
By the second Con-	}	$2$	$a + 3e = x$
dition	-		

Find

Find the Value of  $a$ , from the first Equation.

$$\begin{array}{l|l} 1 - e & 3 \\ 3 \div 2 & 4 \end{array} \left| \begin{array}{l} 2a = b - e \\ a = \frac{b - e}{2} \end{array} \right.$$

Now find the Value of  $a$ , from the second Equation.

$$2 - 3e \mid 5 \mid a = x - 3e$$

Make the fourth and fifth Equations equal to one another, because they are each equal to the same Quantity, which Equation will exterminate  $a$ .

$$\begin{array}{l|l} 4 \cdot 5 & 6 \\ 6 \times 2 & 7 \\ 7 + 6e & 8 \\ 8 - b & 9 \\ 9 \div 5 & 10 \end{array} \left| \begin{array}{l} \frac{b - e}{2} = x - 3e^* \\ b - e = 2x - 6e \\ b + 5e = 2x \\ 5e = 2x - b \\ e = \frac{2x - b}{5} = 6 \text{ Pounds, won by B.} \end{array} \right.$$

Then from the } fifth Equation }  $41 \mid a = x - 3e = 21 \text{ Pounds, won by A.}$

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$$\begin{array}{l} 2a + e = 48 \\ a + 3e = 39 \end{array}$$

Question 46. *What are those two Numbers, that twice the greater being added to three times the lesser, the Sum is 29:*

*And three times the greater being subtracted from five times the lesser, the Remainder is 4.*

Let  $a$  = the greater Number,  $e$  = the lesser Number,  $b = 29$ ,  $m = 4$ .

$$\begin{array}{l} \text{By the first Condition} \\ \text{By the second Condition} \end{array} \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} 2a + 3e = b \\ 5e - 3a = m \end{array}$$

Find

Find the Value of  $a$ , from the first Equation.

$$\begin{array}{l|l|l} 1-3e & 3 & 2a=b-3e \\ 3\div 2 & 4 & a=\frac{b-3e}{2} \end{array}$$

Now find the Value of  $a$ , from the second Equation; transpose  $3a$ , because it has the *negative* Sign.

$$\begin{array}{l|l|l} 2+3a & 5 & 5e=m+3a \\ 5-m & 6 & 5e-m=3a \\ \text{Or} & 7 & 3a=5e-m \\ 7\div 3 & 8 & a=\frac{5e-m}{3} \end{array}$$

Make the fourth and eighth Equations equal to one another, for they are each equal to the same Quantity  $a$ , and this unknown Quantity will be exterminated.

$$\begin{array}{l|l|l} 4 \cdot 8 & 9 & \frac{5e-m}{3} = \frac{b-3e}{2} \\ 9 \times 2 & 10 & \frac{10e-2m}{3} = b-3e \\ 10 \times 3 & 11 & 10e-2m = 3b-9e \\ 11 + 9e & 12 & 19e-2m = 3b \\ 12 + 2m & 13 & 19e = 3b+2m \\ 13 \div 19 & 14 & e = \frac{3b+2m}{19} = 5, \text{ the lesser Number.} \\ \text{By the fourth} & \} & \\ \text{Equation} & \} & 15 \quad a = \frac{b-3e}{2} = 7, \text{ the greater Number.} \end{array}$$

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$$\begin{array}{l} 2a+3e=29 \\ 5e-3a=4 \end{array}$$

Question 47. Two Travellers, A and B, meeting on the Road, found, that if the Number of Miles travelled by A was divided by five, adding to this Quotient three times the Number of Miles travelled by B, the Sum was 249:

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But if twice the Number of Miles travelled by A were added to four times the Number of Miles travelled by B, the Sum was 540. How many Miles had each travelled?

Let  $a$  = the Number of Miles travelled by A,  $e$  = the Number of Miles travelled by B,  $x = 249$ ,  $z = 540$ .

$$\begin{array}{lcl}
 \text{By the first Con-} & \left. \begin{array}{l} \text{dition} \\ - \\ \text{By the second Con-} \\ \text{dition} \end{array} \right\} & \begin{array}{l} 1 \left| \frac{a}{5} + 3e = x \\ 2 \left| 2a + 4e = z \right. \\ \hline 1 \times 5 \quad 3 \left| a + 15e = 5x \\ 3 - 15e \quad 4 \left| a = 5x - 15e \right. \end{array}
 \end{array}$$

The Value of  $a$  being now found by the first Equation, find its Value from the second Equation.

$$\begin{array}{lcl}
 2 - 4e & \left| \begin{array}{l} 5 \\ 5 \div 2 \end{array} \right. & \begin{array}{l} 2a = z - 4e \\ 6 \left| a = \frac{z - 4e}{2} \right. \end{array}
 \end{array}$$

Now make the fourth and sixth Equations equal to one another as before, which exterminates  $a$ .

$$\begin{array}{lcl}
 4 \cdot 6 & \left| \begin{array}{l} 7 \\ 7 \times 2 \\ 8 + 30e \\ 9 - z \\ 10 \div 26 \end{array} \right. & \begin{array}{l} 7 \left| \frac{z - 4e}{2} = 5x - 15e \\ 8 \left| z - 4e = 10x - 30e \\ 9 \left| z + 26e = 10x \\ 10 \left| 26e = 10x - z \\ 11 \left| e = \frac{10x - z}{26} = 75, \text{ the Miles tra-} \right. \right. \right. \\ \text{Then by the 4th} & \left. \left. \left. \begin{array}{l} \text{Equation} \end{array} \right\} \right. \right. & \begin{array}{l} 12 \left| a = 5x - 15e = 120, \text{ the Miles} \right. \\ \text{Equation} & \left. \left. \left. \begin{array}{l} \text{(travelled by B.)} \\ \text{(travelled by A.)} \end{array} \right. \right. \right. \end{array}
 \end{array}$$

P R O O F.

$$\begin{array}{l}
 \frac{a}{5} + 3e = 249 \\
 2a + 4e = 540
 \end{array}$$

The

The Learner being now a little conversant with these Kind of Questions, let the last be repeated, and put Letters for all the Numbers both known and unknown, and if he finds any Difficulty in solving it, by comparing the two Operations, the former may in some Manner explain this; and to illustrate it the more, I have placed the Equations in the last Work, against their correspondend Equations in the next Operation.

Question 48. *Two Travellers, A and B, meeting on the Road, found, that if the Number of Miles travelled by A was divided by 5, and adding to the Quotient 3 times the Miles travelled by B, the Sum was 249:*

*But the Miles travelled by A being multiplied by 2, and added to 4 times the Miles travelled by B, the Sum was 540. How many Miles had each travelled?*

Let  $a$  = the Number of Miles travelled by A,  $e$  = the Number of Miles travelled by B,  $x = 249$ ,  $z = 540$ , as before, but now put  $d = 5$ ,  $m = 3$ ,  $q = 2$ ,  $p = 4$ .

$$\begin{array}{l} \text{By the first Con-} \\ \text{dition} \end{array} \left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} \frac{a}{d} + me = x, \text{ that is, } \frac{a}{5} + 3e = x \\ qa + pe = z, \text{ that is, } 2a + 4e = z \end{array}$$

$$\begin{array}{l} \text{By the second Con-} \\ \text{dition} \end{array} \left. \begin{array}{l} 3 \\ 4 \end{array} \right\} \begin{array}{l} a + dme = dx, \text{ that is, } a + 15e = 5x \\ a = dx - dme, \text{ that is, } a = 5x - 15e \end{array}$$

Having found the Value of  $a$  from the first Equation, find its Value from the second Equation.

$$\begin{array}{l} 2 - pe \\ 5 \div q \end{array} \left| \begin{array}{l} 5 \\ 6 \end{array} \right. \begin{array}{l} qa = z - pe, \text{ that is, } 2a = z - 4e \\ a = \frac{z - pe}{q}, \text{ that is, } a = \frac{z - 4e}{2} \end{array}$$

Now make the fourth and sixth Equations equal to one another, for they are both equal to the same Quantity  $a$ , which exterminates that unknown Quantity.

$$\begin{array}{l} 4 \cdot 6 \\ 7 \times q \end{array} \left| \begin{array}{l} 7 \\ 8 \end{array} \right. \begin{array}{l} \frac{z - pe}{q} = dx - dme, \text{ that is, } \frac{z - 4e}{2} \\ \quad = 5x - 15e \\ z - pe = dqx - dmeq, \text{ that is, } z - 4e \\ \quad = 10x - 30e \end{array}$$

$8 + dmeq$



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$$\begin{array}{l|l} 8 + d m e q & 9 \left| \begin{array}{l} d m e q + z - p e = d q x, \text{ that is, } 26 e \\ + z = 10 x \end{array} \right. \\ 9 - z & 11 \left| \begin{array}{l} d m e q - p e = d q x - z, \text{ that is, } 26 e \\ = 10 x - z. \end{array} \right. \end{array}$$

The unknown Quantity  $e$  being in two Terms, therefore divide by both the Co-efficients of  $e$ , as at Art. 52.

$$\begin{array}{l|l} 10 \div d m q - p & 11 \left| e = \frac{d q x - z}{d m q - p} = 75, \text{ that is, } e = \right. \\ & \left. \frac{10 x - z}{26} = 75, \text{ the Miles travelled} \right. \\ & \left. \text{by B.} \right. \end{array}$$

And it being found that  $e$  is 75, we may find  $a$  by the fourth or sixth Equation to be 120.

And now for the future we shall put Letters for the Numbers that are known, as well as for those that are unknown.

*Question 49. There are two Armies ready to engage; if the Number of Soldiers in both Armies are added together, and that Sum multiplied by 4, the Product is 84440:*

*But if the Number of Men in the greatest Army be multiplied by 2, and added to the Product of the Number of Men in the lesser Army multiplied by 3, the Sum is 52219. To find the Number of Men in each Army?*

Let  $a$  = the Number of Men in the greatest Army,  $e$  = the Number of Men in the lesser Army,  $d = 4$ ,  $m = 84440$ ,  $z = 2$ ,  $x = 3$ ,  $b = 52219$ .

$$\begin{array}{l|l} \text{By the first Con-} & \left. \begin{array}{l} 1 \left| d a + d e = m \\ \text{dition} \right. \end{array} \right\} \\ \text{By the second} & \left. \begin{array}{l} 2 \left| z a + x e = b \\ \text{Condition} \right. \end{array} \right\} \end{array}$$

Find the Value of  $a$ , in the first Equation.

$$\begin{array}{l|l} 1 - d e & 3 \left| \begin{array}{l} d a = m - d e \\ 3 \div d \end{array} \right. \\ 3 \div d & 4 \left| a = \frac{m - d e}{d} \right. \end{array}$$

Now find the Value of  $a$  from the second Equation.

$$\begin{array}{l|l} 2 - x e & 5 \\ 5 \div z & 6 \end{array} \left| \begin{array}{l} z a = b - x e \\ a = \frac{b - x e}{z} \end{array} \right.$$

Make the fourth and fifth Equations equal to one another to exterminate  $a$ .

$$\begin{array}{l|l} 4 \cdot 6 & 7 \\ 7 \times d & 8 \\ 8 \times z & 9 \end{array} \left| \begin{array}{l} \frac{m - d e}{d} = \frac{b - x e}{z} \\ m - d e = \frac{d b - d x e}{z} \\ z m - z d e = d b - d x e \end{array} \right.$$

Now in this Equation  $e$  being on both Sides, find which of its Co-efficients  $d x$  or  $z d$  is the greatest.  $z d$  is 8, but  $d x$  is 12, therefore transpose  $d x e$ , that the unknown Quantity, with the greatest Co-efficient, may have the affirmative Sign, as at Art. 52.

$$\begin{array}{l|l} 9 + d x e & 10 \\ 11 - z m & 11 \\ 11 \div d x - z d & 12 \end{array} \left| \begin{array}{l} d x e + z m - z d e = d b \\ d x e - z d e = d b - z m \\ e = \frac{d b - z m}{d x - z d} = 9999 \end{array} \right.$$

By the sixth Equation } 13  $a = \frac{b - x e}{z} = 11111$ , the Number of Men in the greatest Army.

Dividing the eleventh Equation by  $d x - z d$ , the two Co-efficients of  $e$ , as at Art. 52. gives the twelfth Equation.

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$$\begin{array}{l} 4 a + 4 e = 84440 \\ 2 a + 3 e = 52219 \end{array}$$

Question 50. *A Gentleman bought a Pair of Horses for his Coach, his Son having learnt Algebra, the Father proposed for him to determine the Price of each Horse from saying,*

*That if the Pounds both Horses cost were multiplied by 4, and this Product divided by 8, the Quotient was 20:*

*But*

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But if the Pounds the best Horse cost were multiplied by 3, and this Product added to 5 times the Pounds the worst Horse cost, the Sum was 158 Pounds. Now what was the Price of each Horse?

Let  $a$  = the Pounds the best Horse cost,  $e$  = the Pounds the worst Horse cost,  $b = 4$ ,  $d = 8$ ,  $m = 20$ ,  $p = 3$ ,  $x = 5$ ,  $z = 158$ .

By the first Con- dition	}	1	$\frac{ba + be}{d} = m$	
By the second Condition		2	$pa + xe = z$	
		3	$ba + be = dm$	
$1 \times d$		4	$ba = dm - be$	
$3 - be$		5	$a = \frac{dm - be}{b}$	
$4 \div b$		6	$pa = z - xe$	
$2 - xe$		7	$a = \frac{z - xe}{p}$	
$6 \div p$			To exterminate $a$	
		8	$\frac{dm - be}{b} = \frac{z - xe}{p}$	
$5 \cdot 7$		9	$\frac{pdm - pbe}{b} = z - xe$	
$8 \times p$		10	$pdm - pbe = bz - bxe$	
$9 \times b$		11	$bxe + pdm - pbe = bz$	
$10 + bxe$		12	$bxe - pbe = bz - pdm$	
$11 - pdm$		13	$e = \frac{bz - pdm}{bx - pb} = 19$ Pounds, the Price of the worst Horse.	
$12 \div bx - pb$		14	$a = \frac{z - xe}{p} = 21$ Pounds, the Price of the best Horse.	
By the seventh Equation	}			

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$$\frac{4a + 4e}{8} = 20.$$

$$3a + 5e = 158.$$

Question 51. Two young Gentlemen, who had studied Numbers, not agreeing about their Age, referred the Dispute to their Father, who smiling told them, that if the Age of the eldest was divided

by 2, to which Quotient adding 4 times the Age of the youngest, and extracting the square Root of this Sum, it will be 10:

But if the Age of the eldest was multiplied by 3, and added to the Age of the youngest multiplied by 5, this Sum will be 201. To find the Age of each Person?

Let  $a$  = the Age of the elder,  $e$  = the Age of the younger,  $b = 2$ ,  $d = 4$ ,  $m = 10$ ,  $p = 3$ ,  $z = 5$ ,  $r = 201$ .

By the first Con-	1	$\sqrt{\frac{a}{b} + de} = m$
dition		
By the second	2	$pa + ze = r$
Condition		

Because in the first Equation,  $a$  the unknown Quantity, is under the radical Sign, therefore square both Sides of the Equation, as at Art. 49. The 1 & 2 in the Register signifies that the first Equation being involved or raised to the second Power or Square makes the third Equation, for & is the Sign of Involution.

1 & 2	3	$\frac{a}{b} + de = mm$
3 - de	4	$\frac{a}{b} = mm - de$
4 × b	5	$a = bmm - bde$
2 - ze	6	$pa = r - ze$
6 ÷ p	7	$a = \frac{r - ze}{p}$
		Now to exterminate $a$
5 · 7	8	$\frac{r - ze}{p} = bmm - bde$
8 × p	9	$r - ze = pbmm - pbde$
9 + pbde	10	$pbde + r - ze = pbmm$
10 - r	11	$pbde - ze = pbmm - r$
11 ÷ pbd - z	12	$e = \frac{pbmm - r}{pbd - z} = 21$ , the Age of the (youngest.
By the seventh Step	13	$a = \frac{r - ze}{p} = 32$ , the Age of the (eldest.

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$$\sqrt{\frac{a}{2} + 4e} = 10.$$

$$3a + 5e = 201.$$

Question 52. Two Tradesmen, A and B, comparing their Gains, found, that if the Pounds gained by A were multiplied by 2, to which adding 3 times the Pounds gained by B, the square Root of this Sum was 11 Pounds:

But if 6 times the Pounds gained by B, were added to the Quotient of the Pounds gained by A divided by 10, this Sum was 47 Pounds. To find the Gains of each Tradesman?

Let  $a$  = the Pounds gained by A,  $e$  = the Pounds gained by B,  $b = 2$ ,  $d = 3$ ,  $n = 11$ ,  $p = 6$ ,  $z = 10$ ,  $x = 47$ .

By the first Con-	1	$\sqrt{ba + de} = n$
dition		
By the second	2	$pe + \frac{a}{z} = x$
Condition		

In the first Equation the unknown Quantity  $a$  being under the radical Sign, square both Sides of the Equation as in the last Question.

1 + 2	3	$ba + de = nn$
3 - de	4	$ba = nn - de$
4 ÷ b	5	$a = \frac{nn - de}{b}$
2 × z	6	$zpe + a = zx$
6 - zpe	7	$a = zx - zpe$
5 · 7	8	$\frac{nn - de}{b} = zx - zpe$
8 × b	9	$nn - de = bzx - bzpe$
9 + bzpe	10	$bzpe + nn - de = bzx$
10 - nn	11	$bzpe - de = bzx - nn$

$$\begin{array}{l|l}
 11 \div b z p - d & 12 \\
 \text{By the seventh Step} \} & 13
 \end{array}
 \left| \begin{array}{l}
 e = \frac{b z x - n n}{b z p - d} = 7 \text{ Pounds gained (by B.)} \\
 a = z x - z p e = 50 \text{ Pounds gained (by A.)}
 \end{array} \right.$$

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$$\begin{aligned}
 \sqrt{2a + 3e} &= 11. \\
 6e + \frac{a}{10} &= 47.
 \end{aligned}$$

**Question 53.** *Two Persons, A and B, owe such a Sum of Money, that if the Pounds A owes are divided by 5, to which Quotient adding 4 times the Pounds B owes, and extract the Square Root of this Sum, it will be 6 Pounds:*

*But if from 3 times the Pounds A owes, is subtracted 50 times the Pounds B owes, and extract the Square Root of this Remainder, it will be 10 Pounds. What did each Person owe?*

Let  $a$  = the Pounds A owes,  $e$  = the Pounds B owes,  $m = 5$ ,  $n = 4$ ,  $d = 6$ ,  $p = 3$ ,  $x = 50$ ,  $z = 10$ .

$$\begin{array}{l|l}
 \text{By the first Condition} \} & 1 \\
 \text{By the second Condition} \} & 2
 \end{array}
 \left| \begin{array}{l}
 \sqrt{\frac{a}{m} + n e} = d \\
 \sqrt{p a - x e} = z
 \end{array} \right.$$

To find the Value of  $a$  in the first Equation, raise it to the second Power as in the last Question.

$$\begin{array}{l|l}
 1 \odot 2 & 3 \\
 3 - n e & 4 \\
 4 \times m & 5
 \end{array}
 \left| \begin{array}{l}
 \frac{a}{m} + n e = d d \\
 \frac{a}{m} = d d - n e \\
 a = m d d - m n e
 \end{array} \right.$$

To find the Value of  $a$  in the second Equation, raise it to the second Power as before.

$$\begin{array}{l|l|l} 2 \ominus 2 & 6 & pa - xe = zz \\ 6 + xe & 7 & pa = zz + xe \\ 7 \div p & 8 & a = \frac{zz + xe}{p} \end{array}$$

Now make the fifth and eighth Equations equal to one another to exterminate  $a$ .

$$\begin{array}{l|l|l} 5.8 & 9 & \frac{zz + xe}{p} = mdd - mne \\ 9 \times p & 10 & zz + xe = pmd - pmne \\ 10 + pmne & 11 & pmne + zz + xe = pmd \\ 11 - zz & 12 & pmne + xe = pmd - zz \\ 12 \div pmn + x & 13 & e = \frac{pmd - zz}{pmn + x} = 4 \text{ Pounds, the Debt (of B.)} \\ \text{Then by the eighth Step} & 14 & a = \frac{zz + xe}{p} = 100 \text{ Pounds, the Debt (of A.)} \end{array}$$

# P R O O F.

$$\begin{array}{l} \sqrt{\frac{a}{5} + 4e} = 6. \\ \sqrt{3a - 50e} = 10. \end{array}$$

Question 54. Two Men, A and B, going to Market with Eggs, if the Number of Eggs that A had were multiplied by 6, to which adding 100, and dividing the Sum by the Number of Eggs that B had, the Quotient is 16:

And if from 9 times the Number of Eggs A had, is subtracted 4 times the Number of Eggs B had, there remains 350. How many Eggs had each Person?

Let  $a$  = the Number of Eggs A had,  $e$  = the Number of Eggs B had,  $d = 6$ ,  $m = 100$ ,  $p = 16$ ,  $b = 9$ ,  $x = 4$ ,  $z = 350$ .

$$\begin{array}{l|l|l} 1 & \frac{da + m}{e} = p \\ 2 & ba - xe = z \\ 1 \times e & 3 & da + m = pe \end{array} \left. \vphantom{\begin{array}{l} 1 \\ 2 \\ 3 \end{array}} \right\} \text{By the Question.}$$





Let  $a$  = the Pounds A loft,  $e$  = the Pounds B loft,  $d = 3$ ,  
 $m = 100$ ,  $n = 10$ ,  $x = 250$ ,  $z = 600$ ,  $b = 2$ .

$$\left\{ \begin{array}{l} 1 \left| \sqrt{\frac{da+m}{e}} = n \\ 2 \left| \sqrt{\frac{xe-z}{a}} = b \end{array} \right. \right\} \text{By the Question.}$$

To find the Value of  $a$  in the first Equation, raise it to the second Power by Art. 49.

$$\begin{array}{l|l} 1 \odot 2 & 3 \left| \frac{da+m}{e} = nn \\ 3 \times e & 4 \left| da+m = enn \\ 4 - m & 5 \left| da = enn - m \\ 5 \div d & 6 \left| a = \frac{enn-m}{d} \end{array}$$

To find the Value of  $a$  in the second Equation, raise it to the second Power by Art. 49.

$$\begin{array}{l|l} 2 \odot 2 & 7 \left| \frac{xe-z}{a} = bb \\ 7 \times a & 8 \left| xe-z = abb \\ 8 \div bb & 9 \left| \frac{xe-z}{bb} = a \end{array}$$

Make the sixth and ninth Equations equal to one another, to exterminate  $a$ .

$$\begin{array}{l|l} 6 \cdot 9 & 10 \left| \frac{enn-m}{d} = \frac{xe-z}{bb} \\ 10 \times d & 11 \left| enn-m = \frac{dxe-dz}{bb} \\ 11 \times bb & 12 \left| bbenn-m = dxe-dz \end{array}$$

Because  $dx$ , one Co-efficient of  $e$ , is greater than  $bbnn$ , the other Co-efficient of  $e$ , therefore transpose  $bbnn$ , by Art. 52.

12 — $bbnn e$	13	$-bbm = dx e - dx - bbnn e$
Or	14	$dx e - dz - bbnn e = -bbm$
14 + $dz$	15	$dx e - bbnn e = dz - bbm$
15 ÷ $dx - bbnn$	16	$e = \frac{dz - bbm}{dx - bbnn} = 4$ , the Pounds B (loft.
By the ninth Step	17	$a = \frac{x e - z}{bb} = 100$ , the Pounds A (loft.

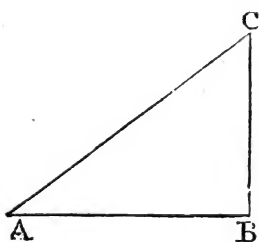
## P R O O F.

$$\sqrt{\frac{3a + 100}{e}} = 10.$$

$$\sqrt{\frac{250e - 600}{a}} = 2.$$

Question 56. In the right-angled Triangle ABC, there is given the Base AB = 4, and the Difference between the Hypothenuse AC and Perpendicular BC = 2. To find the Hypothenuse AC and Perpendicular BC?

Let AC =  $a$ , BC =  $e$ , AB =  $b = 4$ ,  $m = 2$ .



Having put Letters for the three Sides of the Triangle, and amongst these there being two unknown Quantities  $a$  and  $e$ , therefore we must raise two Equations either from the Properties of the Figure, or from the Conditions of the Question. And in the Solution of Geometrical Questions, I would recommend it to the Learner, that after all the Parts of the Figure which

are necessary to the Solution of the Question are expressed by Letters, to observe how many of them are unknown, for generally so many different Equations are raised from the Properties of the Figure, or the Conditions of the Question; afterwards the Work is regulated by the Rules already given.

Now from the Property of the Figure, the Square of the Hypothenuse AC, or  $aa$ , is equal to the Square of the Base AB, or  $bb$ , added to the Square of the Perpendicular BC, or  $ee$ , by 47 *e* 1.

That

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That is  $\left| \begin{array}{l} 1 \\ \dots \end{array} \right| a a = b b + e e$  from the Property of the Figure by 47 e 1.

Because by the Question, the Difference between the Hypotenuse AC, or  $a$ , and Perpendicular BC, or  $e$ , is 2, or  $m$ .

Hence  $\left| \begin{array}{l} 2 \\ \dots \end{array} \right| a - e = m$  by the Conditions of the Question.

Having raised the two Equations, proceed as in the former Examples, that is, first find the Value of  $a$  in the first Equation, by the Extraction of Roots, as at Art. 50.

$$1 \text{ w } 2 \mid 3 \mid a = \sqrt{b b + e e}$$

Now find the Value of  $a$ , in the second Equation.

$$2 + e \mid 4 \mid a = m + e$$

Make the third and fourth Equations equal to one another, to exterminate  $a$ .

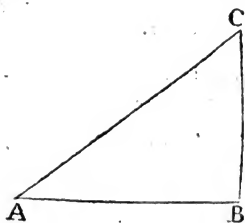
$$3 \cdot 4 \mid 5 \mid m + e = \sqrt{b b + e e}$$

Because  $e$  the unknown Quantity is under the radical Sign, and there being no other Quantities on that Side of the Equation, but what are under the radical Sign, therefore square both Sides of the Equation, as at Art. 49.

$$\begin{array}{l|l} 5 \text{ } 2 & 6 \\ 6 - e e & 7 \\ 7 - m m & 8 \\ 8 \div 2 m & 9 \end{array} \left| \begin{array}{l} m m + 2 m e e + e = b b + e e \\ m m + 2 m e = b b \\ 2 m e = b b - m m \\ e = \frac{b b - m m}{2 m} = 3, \text{ the Perpendicular BC.} \end{array} \right.$$

By the fourth Step  $\left| \begin{array}{l} 10 \\ \dots \end{array} \right| a = m + e = 5$ , the Hypotenuse AC.

To prove these are the three Sides of a right-angled Triangle, square the Hypotenuse 5, and that will be equal to the Square of the Base 4, added to the Square of the Perpendicular 3; for this is the celebrated Property of the right-angled Triangle to have the Square of the Hypotenuse equal to the Sum of the Squares of the Base and Perpendicular.



**Question 57.** In the right-angled Triangle ABC, given the Perpendicular BC = 3, and the Difference between the Hypotenuse AC, and Base AB = 1. To find the Hypotenuse AC, and Base BA?

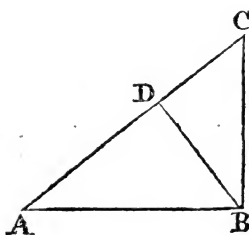
Let AC =  $a$ , BC = 3 =  $b$ ,  
AB =  $e$ ,  $x = 1$ .

Then | 1 |  $aa = bb + ee$ , by the Property of the  
Figure, as in the last Question.  
And | 2 |  $a - e = x$  by the Question.

There being as many Equations raised from the Property of the Figure, and the Conditions of the Question, as there are unknown Quantities, the Work proceeds upon the same general Rules, thus

1 w 2	3	$a = \sqrt{bb + ee}$
2 + e	4	$a = x + e$
3 . 4	5	$x + e = \sqrt{bb + ee}$
5 $\ominus$ 2	6	$xx + 2xe + ee = bb + ee$
6 - ee	7	$xx + 2xe = bb$
7 - xx	8	$2xe = bb - xx$
8 $\div$ 2 x	9	$e = \frac{bb - xx}{2x} = 4$ , the Base AB.

By the fourth Step | 10 |  $a = x + e = 5$ , the Hypotenuse AC.



**Question 58.** In the right-angled Triangle ABC, there is given the Hypotenuse AC = 5, the Base AB = 4, and the Perpendicular BC = 3, to find the Perpendicular BD, let fall from the Angle B, upon the Hypotenuse AC.

Let AC =  $b = 5$ , AB =  $m = 4$ ,  
BC =  $x = 3$ , DC =  $a$ , AD =  $e$ .

The

The Question requiring that we find BD, if we find CD we can answer the Question, for the Triangle BDC being a right-angled Triangle, BD being perpendicular to AC, consequently BC being known, and by finding DC, we shall afterwards easily find DB, by the common Property of the Triangle.

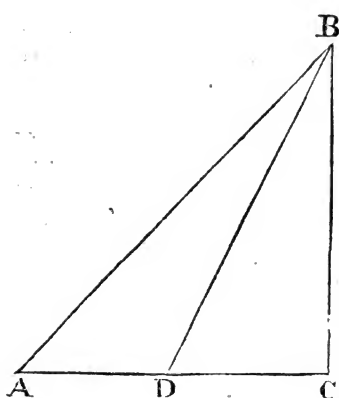
It is exactly the same, if we find AD, for the Triangle ADB is right-angled, and AB is given by the Question.

Now BD being a Perpendicular common to the two Triangles ABD, and BDC, let  $BD = p$ , then from the right-angled Triangle ABD, we have  $mm - ee = pp$ , and by the right-angled Triangle CBD, we have  $xx - aa = pp$ , from the same Reasoning as in the two last Questions.

Consequently	1	$mm - ee = xx - aa$ , for both $mm - ee$ and $xx - aa$ , are equal to the same Quantity $pp$ , and therefore equal to one another.
And	2	$a + e = b$ , that is, $AD + DC = AC$ by the Figure. To find the Value of $a$ in the first Equation.
$1 + aa$	3	$aa + mm - ee = xx$
$3 + ee$	4	$aa + mm = xx + ee$
$4 - mm$	5	$aa = xx + ee - mm$
$5 \text{ w } 2$	6	$a = \sqrt{xx + ee - mm}$ Now find the Value of $a$ in the second Equation.
$2 - e$	7	$a = b - e$
$6 . 7$	8	$\sqrt{xx + ee - mm} = b - e$
$8 \oplus 2$	9	$xx + ee - mm = bb - 2be + ee$
$9 - ee$	10	$xx - mm = bb - 2be$
$10 + 2be$	11	$2be + xx - mm = bb$
$11 + mm$	12	$2be + xx = bb + mm$
$12 - xx$	13	$2be = bb + mm - xx$
$13 \div 2b$	14	$e = \frac{bb + mm - xx}{2b} = 3.2 = AD.$

Having found AD to be 3.2 it will be easy to find DB by what was said above. Thus,

$$\begin{array}{r}
 16 \text{ the Square of } AB. \\
 - 10.24 \text{ the Square of } AD. \\
 \hline
 5.76 \text{ (2.4 = DB, the Perpendicular required.)} \\
 4 \\
 \hline
 44) 176 \\
 \underline{176} \\
 0
 \end{array}$$



Question 59. In the oblique Triangle ADB, there is given the Side  $AB = 15$ , the Side  $BD = 12$ , and the Side  $AD = 6$ , to find the Perpendicular  $BC$  falling without the Triangle from the Angle  $B$ , on the Side  $AD$ , continued.

This Question will be answered from finding  $DC$ , for the Triangle  $BCD$  being right-angled, and  $DB$  being known from finding  $DC$ , we may then find  $BC$  from the common Property of the Triangle  $DBC$ , as in the last Question.

Let  $AB = b = 15$ ,  $AD = m = 6$ ,  $DB = x = 12$ ,  $DC = a$ , then  $AC = AD + DC = m + a$ ,  $BC = e$ .

Because the Triangle  $ABC$  is right-angled, therefore if from the Square of  $AB$ , or  $bb$ , we subtract the Square of  $AC$ , or  $mm + 2ma + aa$ , the Remainder is equal to the Square of  $CB$ , or  $ee$ .

$$\text{Therefore } | 1 | bb - mm - 2ma - aa = ee.$$

Because the Triangle  $DBC$  is right-angled, by the same Reasoning we have

$$\text{Again } | 2 | xx - aa = ee.$$

And

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And as the first and second Equations are each = *ee*, therefore make them equal to one another, which exterminates every Power of *e* in those Equations.

$$\begin{array}{r|l}
 1 \cdot 2 & 3 \quad x x - a a = b b - m m - 2 m a - a a \\
 3 + a a & 4 \quad x x = b b - m m - 2 m a \\
 4 + 2 m a & 5 \quad 2 m a + x x = b b - m m \\
 5 - x x & 6 \quad 2 m a = b b - m m - x x \\
 6 \div 2 m & 7 \quad a = \frac{b b - m m - x x}{2 m} = 3.75 = D C.
 \end{array}$$

And from hence we may find B C as was said above, thus

$$\begin{array}{r}
 B D, \text{ or } x = 12 \\
 \underline{12} \\
 144
 \end{array}$$

$$\begin{array}{r}
 D C, \text{ or } a = 3.75 \\
 \underline{3.75} \\
 1875 \\
 2625 \\
 \underline{1125} \\
 14.0625
 \end{array}$$

$$\begin{array}{r}
 144. \\
 \underline{14.0625} \\
 129.9375 \text{ (11.39 = BC, the Perpendicular required.)} \\
 \underline{1} \\
 21) 29 \\
 \underline{21} \\
 223) 893 \\
 \underline{669} \\
 2269) 22475 \\
 \underline{20421} \\
 2054
 \end{array}$$

## Of Quadratic EQUATIONS.

56. **W**HEN all the known Quantities are on one Side of the Equation, and those Quantities only on the other Side which have some Power of the unknown Quantity; then if the unknown Quantity appears to be to the *second Power* or *Square* in one Term, and to the *first Power* only in another Term; or if in one Term, its Power or Height is *double* its Power or Height in another Term, and there is no *other* Power of the unknown Quantity in the Equation, these Equations are called *Quadratic*, as in the following Questions.

Question 60. *Two Men had such a Number of Shillings, that the lesser being subtracted from the greater, there remains 10:*

*But the Number of Shillings one Man had multiplied by the Number of Shillings the other Man had, the Product is 75. To find each Man's Number of Shillings?*

Let  $a$  = the greater Number of Shillings one of the Men had,  $e$  = the lesser Number of Shillings the other Man had,  $b = 10$ ,  $m = 75$ .

Then	1	$a - e = b$	} By the Question,
And	2	$ae = m$	
$1 + e$	3	$a = b + e$	
$2 \div e$	4	$a = \frac{m}{e}$	
$3 \cdot 4$	5	$e + b = \frac{m}{e}$	
$5 \times e$	6	$ee + be = m$	

From comparing the sixth Equation with what is said above, it appears to be *Quadratic*, for one Quantity is  $ee$ , or  $e$  to the second Power, and in the other Quantity it is only  $e$ , or  $e$  to the first Power.

And



# Of Quadratic EQUATIONS. 169

And to resolve this Equation, take  $b$  the Co-efficient of  $e$  to the first Power, and divide it by 2, the Quotient is  $\frac{b}{2}$ , which square or multiply by itself, and the Product is  $\frac{bb}{4}$ , which add to both Sides of the Equation, thus

$$6c \square \mid 7 \mid ee + be + \frac{bb}{4} = m + \frac{bb}{4}$$

The  $c \square$  in the *Register* signifies, that the sixth Step is made a Square at the seventh Step, or the *Square is completed*.

Now if we compare the Side of the Equation  $ee + be + \frac{bb}{4}$ , with some of the Examples at Art. 34. we shall find it to be a *rational* Quantity, or a Square, therefore extract the square Root of both Sides of the Equation :

$$\begin{array}{l} 7 \text{ w } 2 \mid 8 \mid e + \frac{b}{2} = \sqrt{m + \frac{bb}{4}} \\ 8 - \frac{b}{2} \mid 9 \mid e = \sqrt{m + \frac{bb}{4}} : - \frac{b}{2} \end{array}$$

In Numbers.

$$\begin{array}{r} m = 75 \\ bb \\ \hline 4 = 25 \end{array}$$

100 the square Root of which is 10

$$- 5 = - \frac{b}{2}$$

5 =  $e$ , the Number  
of Shillings one of the Men had.

Then by the fourth Step  $a = \frac{m}{e} = 15$ , the Number of Shillings the other Man had.

Z

P R O O F.

## P R O O F.

$$a - e = 10$$

$$ae = 75$$

Question 61. *There are two Numbers, if the Square of the lesser is taken from the greater, there remains 36:*

*But the greater being added to 6 times the lesser, the Sum is 148. What are the two Numbers?*

Let  $a$  = the greater Number,  $e$  = the lesser Number,  
 $b = 36$ ,  $m = 6$ ,  $x = 148$ .

Then	1	$a - ee = b$	} By the Question.
And	2	$a + me = x$	
$1 + ee$	3	$a = b + ee$	
$2 - me$	4	$a = x - me$	
$3 \cdot 4$	5	$b + ee = x - me$	
$5 - b$	6	$ee = x - me - b$	
$6 + me$	7	$ee + me = x - b$	

The unknown Quantities being brought on one Side of the Equation, the Equation appears to be *Quadratic*, by Art. 56.

Now the Co-efficient of the first Power of  $e$  is  $m$ , which divided by 2 is  $\frac{m}{2}$ , this squared is  $\frac{mm}{4}$ , and adding  $\frac{mm}{4}$  to both Sides of the Equation as in the last Question, we have

$$7c \square \mid 8 \mid ee + me + \frac{mm}{4} = x - b + \frac{mm}{4}$$

The  $7c \square$  signifies that the seventh Equation is made a compleat Square, at the eighth Step.

And extracting the Roots of both Sides of the Equation, as in the last Question,

# Of Quadratic EQUATIONS. 171

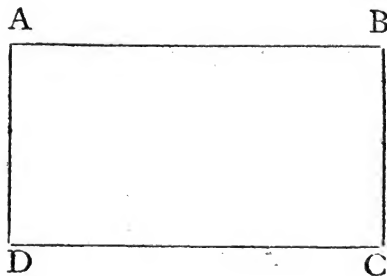
$$\begin{array}{l|l|l}
 8 \text{ or } 2 & 9 & e + \frac{m}{2} = \sqrt{x - b + \frac{m m}{4}} \\
 9 - \frac{m}{2} & 10 & e = \sqrt{x - b + \frac{m m}{4}} : - \frac{m}{2} = 8, \\
 & & \text{(the lesser Number.} \\
 \text{By the fourth Step} & 11 & a = x - m e = 100, \text{ the greater Num-} \\
 & & \text{(ber.)}
 \end{array}$$

## PROOF.

$$\begin{aligned}
 a - e e &= 36 \\
 a + 6 e &= 148
 \end{aligned}$$

Question 62. In the Parallelogram ABCD, if from the longest Side AB multiplied by 3, is subtracted the Square of the shortest Side BC, the Remainder will be 5:

But if the longest Side AB is added to 4 times the shortest Side BC, the Sum is 30. To find the Sides of the Parallelogram AB, and BC?



Let  $a = AB$ ,  $BC = e$ ,  $d = 3$ ,  $m = 5$ ,  $z = 4$ ,  $x = 30$ .

$$\begin{array}{l|l|l}
 & 1 & da - ee = m \\
 & 2 & a + ze = x \\
 1 + ee & 3 & da = m + ee \\
 3 \div d & 4 & a = \frac{m + ee}{d} \\
 2 - ze & 5 & a = x - ze \\
 & & Z 2
 \end{array} \quad \text{By the Question.}$$

$$\begin{array}{r|l}
 4 \cdot 5 & 6 \\
 6 \times d & 7 \\
 7 - m & 8 \\
 8 + dze & 9
 \end{array}
 \left| \begin{array}{l}
 \frac{m + ee}{d} = x - ze \\
 m + ee = dx - dze \\
 ee = dx - dze - m \\
 ee + dze = dx - m
 \end{array} \right.$$

Now the Equation appears to be *Quadratic* by Art. 56. and the Co-efficient of  $e$  is  $dz$ , which divided by 2, is  $\frac{dz}{2}$ , this squared is  $\frac{ddzz}{4}$ , which added to both Sides of the Equation, as in the two last Examples, we have

$$9c \square \left| 10 \right| ee + dze + \frac{ddzz}{4} = dx - m + \frac{ddzz}{4}$$

And extracting the Roots of both Sides of the Equation, as in the two last Questions,

$$\begin{array}{r|l}
 10 \text{ us } 2 & 11 \\
 11 - \frac{dz}{2} & 12
 \end{array}
 \left| \begin{array}{l}
 e + \frac{dz}{2} = \sqrt{dx - m + \frac{ddzz}{4}} \\
 e = \sqrt{dx - m + \frac{ddzz}{4}} : -\frac{dz}{2} = 5 \\
 \hspace{10em} (= BC.
 \end{array} \right.$$

From the fifth Step  $13 \left| a = x - ze = 10 = AB. \right.$

### P R O O F.

$$\begin{array}{l}
 3a - ee = 5 \\
 a + 4e = 30
 \end{array}$$

Question 63. Two Gentlemen having had their Parks surveyed, had lost the Account, but remembered, that if the Number of Acres in A's Park was added to the Number of Acres in B's Park, the Sum was 110:

But if the Number of Acres in B's Park was multiplied by 80, from which Product subtracting the Square of the Number of Acres in A's Park, there remained 400. How many Acres was there in each Park?

Let

# Of Quadratic EQUATIONS. 173

Let  $a$  = the Number of Acres in A's Park,  $e$  = the Number of Acres in B's Park,  $b = 110$ ,  $m = 80$ ,  $x = 400$ .

	1	$a + e = b$	2	$me - aa = x$	} By the Question.
	2	$me - aa = x$	3	$e = b - a$	
$1 - a$	3	$e = b - a$	4	$me = x + aa$	
$2 + aa$	4	$me = x + aa$	5	$e = \frac{x + aa}{m}$	
$4 \div m$	5	$e = \frac{x + aa}{m}$	6	$\frac{x + aa}{m} = b - a$	
$3 \cdot 5$	6	$\frac{x + aa}{m} = b - a$	7	$x + aa = mb - ma$	
$6 \times m$	7	$x + aa = mb - ma$	8	$aa = mb - ma - x$	
$7 - x$	8	$aa = mb - ma - x$	9	$aa + ma = mb - x$	
$8 + ma$	9	$aa + ma = mb - x$			

Here the Equation appears *Quadratic*, and completing the Square as in the former Examples, we have

$$9 \text{ c } \square \quad | \quad 10 \quad | \quad aa + ma + \frac{mm}{4} = mb - x + \frac{mm}{4}$$

And extracting the square Roots of both Sides of the Equation, as in the former Examples,

10 w 2	11	$a + \frac{m}{2} = \sqrt{mb - x + \frac{mm}{4}}$
$11 - \frac{m}{2}$	12	$a = \sqrt{mb - x + \frac{mm}{4}} - \frac{m}{2} = 60,$
		(the Number of Acres in A's Park.
From the third Step	13	$e = b - a = 50,$ the Number of Acres (in B's Park.

P R O O F.

$$\begin{aligned} a + e &= 110 \\ 80e - aa &= 400 \end{aligned}$$

*The Manner of substituting one Quantity for several others explained.*

57. But if, after the Work is prepared for having the Square compleated, it appears that the first Power of the unknown Quantity is in more Terms than one, it will be more convenient to *substitute* some other Letter, for the Co-efficients of the first Power of the unknown Quantity, as in the following Examples.

Question 64. *A Gentleman proposed to give his two Sons, A and B, each an Estate, on the Condition, they could tell him what were their Rents, by knowing, that if the Square of the Rent of the Estate he intended to give A was added to the same Rent multiplied by 7, and the Sum added to the Rent of the Estate he intended to give B, when multiplied by 4, this Sum would be 4220 Pounds :*

*But if the Sum of the Rents of the two Estates was divided by 10, the Quotient was 11 Pounds. What was the Rent of each Estate ?*

Let  $a$  = the Rent of the Estate A was to have,  $e$  = the Rent of the Estate B was to have,  $b = 7$ ,  $m = 4$ ,  $d = 4220$ ,  $p = 10$ ,  $x = 11$ .

$$\left\{ \begin{array}{l} 1 \quad a a + b a + m e = d \\ 2 \quad \frac{a + e}{p} = x \end{array} \right\} \text{By the Question.}$$

These being the two Equations which arise from the Question, and because the Terms are more simple that have the unknown Quantity  $e$ , than those that have the unknown Quantity  $a$ , it may be more convenient to find the Value of  $e$ , in each of the two given Equations. This Caution the Learner may observe for the future, to find the Value of that unknown Quantity whose Terms are the most simple in the given Equations; and those may be taken for the more simple, whose Powers are the lowest in both the Equations that arise from the Question; thus, if one of the unknown Quantities is only to the *first* Power in both the given Equations, when the other unknown Quantity is to the *second* Power in one of the given Equations, the Terms of the former may be said to be more simple, and there-

therefore best to find the Value of that unknown Quantity: The Reader will find this Method observed in the following Questions, and comparing their Work with what is said may make this Direction more intelligible.

$$\begin{array}{r|l}
 1 - ba & 3 \quad aa + me = d - ba \\
 3 - aa & 4 \quad me = d - ba - aa \\
 4 \div m & 5 \quad e = \frac{d - ba - aa}{m} \\
 2 \times p & 6 \quad a + e = px \\
 6 - a & 7 \quad e = px - a \\
 5 \cdot 7 & 8 \quad px - a = \frac{d - ba - aa}{m} \\
 8 \times m & 9 \quad mpx - ma = d - ba - aa \\
 9 + aa & 10 \quad aa + mpx - ma = d - ba \\
 10 + ba & 11 \quad aa + ba + mpx - ma = d \\
 11 - mpx & 12 \quad aa + ba - ma = d - mpx
 \end{array}$$

Here the Equation appears to be Quadratic, and the first Power of  $a$  is in two Terms, viz.  $ba$  and  $ma$ , the two Coefficients being  $b$  and  $m$ , and connected by the Sign  $-$ .

But  $b$  and  $m$ , being known Quantities, therefore  $b - m = 7 - 4 = 3$ , now substitute, or put  $z = 3$ , or  $z = b - m$ , then the last Equation is,

By Substitution | 13 |  $aa + za = d - mpx$ ; for by Substitution  $za = ba - ma$ , and therefore in the room of  $ba - ma$ , we use only  $za$ . Now taking  $z$  for the Co-efficient of  $a$ , and completing the Square as before,

$$\begin{array}{r|l}
 13 \text{ c } \square & 14 \quad aa + za + \frac{zz}{4} = d - mpx + \frac{zz}{4} \\
 14 \text{ un } 2 & 15 \quad a + \frac{z}{2} = \sqrt{d - mpx + \frac{zz}{4}} \\
 15 - \frac{z}{2} & 16 \quad a = \sqrt{d - mpx + \frac{zz}{4}} - \frac{z}{2} = 60,
 \end{array}$$

the Rent of the Estate which A was to have.

By the seventh Step } 17  $e = px - a = 50$ , the Rent of the (Estate which B was to have.

P R O O F.

## PROOF.

$$aa + 7a + 4e = 4220.$$

$$\frac{a+e}{10} = 11.$$

It may be just observed to the Learner, that the Method of *Substitution* is only to save Trouble and Labour, for after the twelfth Step, if we had not substituted  $b - m = x$ , then to have completed the Square, we must have divided  $b - m$  the two Co-efficients of  $a$  by 2, the Quotient of which is  $\frac{b-m}{2}$ , which

squared is  $\frac{bb - 2bm + mm}{4}$ , and this must have been added to both Sides of the Equation, whereas by substituting  $b - m = x$ , the Quantity to be added on both Sides of the Equation is only  $\frac{xx}{4}$ .

Question 65. A Draper bought a Parcel of Linen, and a Parcel of Woollen Cloth, if the Square of the Pounds he gave for the Linen Cloth be divided by 4, and to this Quotient there is added the Pounds each Sort cost, the Sum is 1000 Pounds:

But if the Pounds the Linen cost is added to the Quotient of the Pounds the Woollen cost, divided by 8, the Sum is 65 Pounds. How much was given for each Sort?

Let  $a$  = the Pounds the Linen cost,  $e$  = the Pounds the Woollen cost,  $b = 4$ ,  $d = 1000$ ,  $m = 8$ ,  $x = 65$ .

$$\left. \begin{array}{l|l} 1 & \frac{aa}{b} + a + e = d \\ 2 & a + \frac{e}{m} = x \end{array} \right\} \text{By the Question.}$$


---


$$\begin{array}{r|l} 1 - a & 3 \quad \frac{aa}{b} + e = d - a \\ 3 - \frac{aa}{b} & 4 \quad e = d - a - \frac{aa}{b} \end{array}$$





## PROOF.

$$\frac{aa}{4} + a + e = 1000.$$

$$a + \frac{e}{8} = 65.$$

*To resolve a Quadratic Equation when the Square of the unknown Quantity has a Co-efficient.*

58. But if the Square of the unknown Quantity has any Co-efficient besides Unity, or 1, then before you begin to complete the Square, divide every Term in the Equation by that Co-efficient, after which complete the Square, and proceed as before.

Question 66. *To find two Numbers, that the Square of the greater being multiplied by 4, if this Product is added to 3 times the lesser, the Sum shall be 1606:*

*But if 5 times the greater is added to 6 times the lesser, the Sum shall be 112.*

Let  $a$  = the greater Number,  $e$  = the lesser Number,  $b = 4$ ,  $d = 3$ ,  $m = 1606$ ,  $p = 5$ ,  $x = 6$ ,  $z = 112$ .

	1	$baa + de = m$	} By the Question.
	2	$pa + xe = z$	
$1 - baa$	3	$de = m - baa$	
$3 \div d$	4	$e = \frac{m - baa}{d}$	
$2 - pa$	5	$xe = z - pa$	
$5 \div x$	6	$e = \frac{z - pa}{x}$	
$4 \cdot 6$	7	$\frac{z - pa}{x} = \frac{m - baa}{d}$	
$7 \times x$	8	$z - pa = \frac{xm - xbaa}{d}$	

$$\begin{array}{l|l} 8 \times d & 9 \\ 9 + x b a a & 10 \\ 10 - dz & 11 \end{array} \left| \begin{array}{l} dz - d p a = x m - x b a a \\ x b a a + dz - d p a = x m \\ x b a a - d p a = x m - dz \end{array} \right.$$

The Equation appearing to be *Quadratic*, and all the known Quantities, except those which contain the unknown one, being on one Side of the Equation, and the highest Power of the unknown Quantity having a Co-efficient, divide by that Co-efficient.

$$11 \div x b \left| 12 \right| a a - \frac{d p a}{x b} = \frac{x m - d z}{x b}$$

To avoid the Trouble of dividing  $\frac{d p}{x b}$ , the Co-efficient of  $a$ , by 2, and squaring the Quotient, and adding it to both Sides of the Equation to complet the Square, as in the former Questions, substitute  $-r = -\frac{d p}{x b} = -.625$  then,

$$\begin{array}{l|l} \text{By Substitution} & 13 \\ 13 \text{ c } \square & 14 \end{array} \left| \begin{array}{l} a a - r a = \frac{x m - d z}{x b} \\ a a - r a + \frac{r r}{4} = \frac{x m - d z}{x b} + \frac{r r}{4} \end{array} \right.$$

Now extracting the square Root as in the last Question,

$$\begin{array}{l|l} 14 \text{ w } 2 & 15 \\ 15 + \frac{r}{2} & 16 \end{array} \left| \begin{array}{l} a - \frac{r}{2} = \sqrt{\frac{x m - d z}{x b} + \frac{r r}{4}} \\ a = \sqrt{\frac{x m - d z}{x b} + \frac{r r}{4}} + \frac{r}{2} = 20, \\ \text{(the greater Number.)} \\ \text{By the sixth Step} & 17 \end{array} \right| e = \frac{z - p a}{x} = 2, \text{ the lesser Number.}$$

P R O O F.

$$\begin{array}{l} 4 a a + 3 e = 1606. \\ 5 a + 6 e = 112. \end{array}$$

Question 67. Two Gamesters, A and B, losing at the Gaming-Tables, upon comparing their Losses, found, that if the Square of the Pounds A lost was multiplied by 5, and this Product added to 6 times the Pounds B lost, the Sum was 548 Pounds :

But if the Pounds A lost was multiplied by 3, and to this Product adding the Pounds B lost multiplied by 2, the Sum was 46 Pounds. To find the Loss of each?

Let  $a$  = the Pounds A lost,  $e$  = the Pounds B lost,  $x = 5$ ,  $m = 6$ ,  $d = 548$ ,  $b = 3$ ,  $z = 2$ ,  $r = 46$ .

	1	$x a a + m e = d$	} By the Question.
	2	$b a + z e = r$	
1 — $x a a$	3	$m e = d - x a a$	
3 ÷ $m$	4	$e = \frac{d - x a a}{m}$	
2 — $b a$	5	$z e = r - b a$	
5 — $z$	6	$e = \frac{r - b a}{z}$	
4 . 6	7	$\frac{r - b a}{z} = \frac{d - x a a}{m}$	
7 × $m$	8	$\frac{r m - m b a}{z} = d - x a a$	
8 × $z$	9	$r m - m b a = z d - z x a a$	
9 + $z x a a$	10	$z x a a + r m - m b a = z d$	
10 — $r m$	11	$z x a a - m b a = z d - r m$	

The Equation being *Quadratic*, and all those Terms which contain any Power of  $a$  being on one Side of the Equation, divide by the Co-efficient of its highest Power.

11 ÷ $z x$	12	$a a - \frac{m b a}{z x} = \frac{z d - r m}{z x}$
		Substitute — $p = -\frac{m b}{z x} = -1.8$
Ey Substitution	13	$a a - p a = \frac{z d - r m}{z x}$
13 c □	14	$a a - p a + \frac{p p}{4} = \frac{z d - r m}{z x} + \frac{p p}{4}$

$$\begin{array}{l|l}
 14 \text{ um } 2 & 15 \quad a - \frac{p}{2} = \sqrt{\frac{z d - r m}{z x} + \frac{p p}{4}} \\
 15 + \frac{p}{2} & 16 \quad a = \sqrt{\frac{z d - r m}{z x} + \frac{p p}{4}} + \frac{p}{2} = 10 \\
 & \quad \text{(Pounds, the Sum lost by A.} \\
 \text{By the sixth Step} & 17 \quad e = \frac{r - b a}{z} = 8 \text{ Pounds, the Sum} \\
 & \quad \text{(lost by B.)}
 \end{array}$$

P R O O F.

$$\begin{array}{l}
 5 a a + 6 e = 548 \\
 3 a + 2 e = 46
 \end{array}$$

Question 68. *Two Brothers, A and B, trying each other's Skill in Algebra, says the eldest Brother, the Sum of our Ages is 45:*

*But, says the youngest, if they are multiplied together, the Product is 500. What is the Age of each of them?*

Let  $a$  = the Age of the eldest,  $e$  = the Age of the youngest,  $s = 45$ ,  $p = 500$ .

$$\begin{array}{l|l}
 1 & a + e = s \\
 2 & a e = p \\
 1 - e & 3 \quad a = s - e \\
 2 - e & 4 \quad a = \frac{p}{e} \\
 3 \cdot 4 & 5 \quad \frac{p}{e} = s - e \\
 5 \times e & 6 \quad p = s e - e e
 \end{array} \quad \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \text{By the Question.}$$

Because the Square of the unknown Quantity has the Sign —, therefore transpose it, that the highest Power of the unknown Quantity may have the affirmative Sign.

$$\begin{array}{l|l}
 6 + e e & 7 \quad e e + p = s e \\
 7 - p & 8 \quad e e = s e - p \\
 8 - s e & 9 \quad e e - s e = -p \\
 9 e \square & 10 \quad e e - s e + \frac{s s}{4} = \frac{s s}{4} - p
 \end{array}$$

Question 67. Two Gamesters, A and B, losing at the Gaming-Tables, upon comparing their Losses, found, that if the Square of the Pounds A lost was multiplied by 5, and this Product added to 6 times the Pounds B lost, the Sum was 548 Pounds:

But if the Pounds A lost was multiplied by 3, and to this Product adding the Pounds B lost multiplied by 2, the Sum was 46 Pounds. To find the Loss of each?

Let  $a$  = the Pounds A lost,  $e$  = the Pounds B lost,  $x = 5$ ,  $m = 6$ ,  $d = 548$ ,  $b = 3$ ,  $z = 2$ ,  $r = 46$ .

	1	$x a a + m e = d$	} By the Question.
	2	$b a + z e = r$	
1 — $x a a$	3	$m e = d - x a a$	
3 ÷ $m$	4	$e = \frac{d - x a a}{m}$	
2 — $b a$	5	$z e = r - b a$	
5 — $z$	6	$e = \frac{r - b a}{z}$	
4 . 6	7	$\frac{r - b a}{z} = \frac{d - x a a}{m}$	
7 × $m$	8	$\frac{r m - m b a}{z} = d - x a a$	
8 × $z$	9	$r m - m b a = z d - z x a a$	
9 + $z x a a$	10	$z x a a + r m - m b a = z d$	
10 — $r m$	11	$z x a a - m b a = z d - r m$	

The Equation being *Quadratic*, and all those Terms which contain any Power of  $a$  being on one Side of the Equation, divide by the Co-efficient of its highest Power.

11 ÷ $z x$	12	$a a - \frac{m b a}{z x} = \frac{z d - r m}{z x}$
		Substitute — $p = -\frac{m b}{z x} = -1.8$
By Substitution	13	$a a - p a = \frac{z d - r m}{z x}$
13 c □	14	$a a - p a + \frac{p p}{4} = \frac{z d - r m}{z x} + \frac{p p}{4}$

$$\begin{array}{l|l}
 14 \text{ un } 2 & 15 \quad a - \frac{p}{2} = \sqrt{\frac{z d - r m}{z x} + \frac{p p}{4}} \\
 15 + \frac{p}{2} & 16 \quad a = \sqrt{\frac{z d - r m}{z x} + \frac{p p}{4}} + \frac{p}{2} = 10 \\
 & \quad \text{(Pounds, the Sum lost by A.} \\
 \text{By the sixth Step} & 17 \quad e = \frac{r - b a}{z} = 8 \text{ Pounds, the Sum} \\
 & \quad \text{(lost by B.)}
 \end{array}$$

P R O O F.

$$\begin{array}{l}
 5 a a + 6 e = 548 \\
 3 a + 2 e = 46
 \end{array}$$

Question 68. Two Brothers, A and B, trying each other's Skill in Algebra, says the eldest Brother, the Sum of our Ages is 45:

But, says the youngest, if they are multiplied together, the Product is 500. What is the Age of each of them?

Let  $a$  = the Age of the eldest,  $e$  = the Age of the youngest,  $s = 45$ ,  $p = 500$ .

$$\begin{array}{l|l}
 1 & a + e = s \\
 2 & a e = p \\
 1 - e & 3 \quad a = s - e \\
 2 \div e & 4 \quad a = \frac{p}{e} \\
 3 \cdot 4 & 5 \quad \frac{p}{e} = s - e \\
 5 \times e & 6 \quad p = s e - e e
 \end{array} \quad \left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \text{By the Question.}$$

Because the Square of the unknown Quantity has the Sign —, therefore transpose it, that the highest Power of the unknown Quantity may have the affirmative Sign.

$$\begin{array}{l|l}
 6 + e e & 7 \quad e e + p = s e \\
 7 - p & 8 \quad e e = s e - p \\
 8 - s e & 9 \quad e e - s e = -p \\
 9 e \square & 10 \quad e e - s e + \frac{s s}{4} = \frac{s s}{4} - p
 \end{array}$$

$$\begin{array}{l|l|l}
 10 \text{ or } 2 & 11 & e - \frac{s}{2} = \sqrt{\frac{ss}{4} - p} \\
 11 + \frac{s}{2} & 12 & e = \frac{s}{2} + \sqrt{\frac{ss}{4} - p} = 25, \text{ the Age} \\
 & & \text{(of the youngest.} \\
 \text{By the third Step} & 13 & a = s - e = 20, \text{ the Age of the eldest.}
 \end{array}$$

This Answer to the Question contains an *Absurdity*, for  $e$  that is put for the Age of the *youngest* Brother is 25, when  $a$  that is put for the Age of the *eldest* Brother is only 20.

### *The two Roots of Quadratic Equations explained.*

59. And now we shall explain to the young *Analyst*, that in every *Quadratic* Equation, the unknown Quantity has two Values or Roots, sometimes one is *affirmative*, and the other *negative*, and sometimes both are *affirmative*.

There are *three Forms* of Quadratic Equations.

The first is the sixth Step of Question 60, where we have  $ee + be = m$ .

And of this *Form* are the Equations at Question 61, Step 7. Question 62, Step 9. Question 63, Step 9. Question 64, Step 12.

The second *Form* is the twelfth Step of Question 65, where we have  $aa - za = bd - bmx$ .

And of this *Form* are the Equations at Question 66, Step 11. Question 67, Step 13.

The Difference between these two *Forms* of Quadratic Equations, is only in the lowest Power of the unknown Quantity having the Sign  $+$  or  $-$ , for in the first *Form* it has the Sign  $+$ , it being  $be$ , but in the second *Form* it has the Sign  $-$ , for it is  $-za$ . And if the lowest Power of the unknown Quantity has several Co-efficients connected by the Signs  $+$  or  $-$ , as at Question 64, Step 12. Question 65, Step 11. Then if the Sum of the *positive* or *affirmative* Co-efficients exceeds the Sum of the *negative* Co-efficients, the Equation is of the first *Form*: But, on the contrary, if the Sum of the *negative* Co-efficients exceeds the Sum of the *positive* or *affirmative* Co-efficients, then the Equation is of the second *Form*.



But the third *Form* is the ninth. Step of the last Question, where we have  $ee - se = -p$ , which differs from the other two *Forms* of Quadratic Equations, in this, that if the Side of the Equation, which is known, consists but of one Quantity, as in the present Case, it has the Sign  $-$ ; and if that Side of the Equation consists of several known Quantities connected by the Signs  $+$  or  $-$ , that then the Sum of the *negative* Quantities is always greater than the Sum of the *affirmative* Quantities; but in the first and second *Form*, if there is but one known Quantity, which composes that Side of the Equation, it will always have the *affirmative* Sign; and if there are several known Quantities connected by the Signs  $+$  or  $-$ , that then the Sum of the *affirmative* will always exceed the Sum of the *negative* Quantities.

Now of the two Values or Roots of  $a$  in the first and second *Form* of Quadratic Equations, one is *affirmative*, and the other *negative*; and as the negative Value in these Equations does not come out in the Operation without a Mistake in the Work, therefore these two *Forms* of Quadratic Equations give the true Numbers required.

But the two Values or Roots of  $a$  in the third *Form* are both *affirmative*, and the Answer sometimes giving one, and sometimes the other Root, and it being doubtful in many Cases which of these two Values of  $a$  will answer the Conditions of the Question; this *Form* of Quadratic Equations is therefore called the *Ambiguous Form*.

Before we show the Reason of these two Values or Roots of the unknown Quantity in Quadratic Equations, and how from having found one Number, or Value, the Learner may find the other Number; we shall explain the Division in *Algebra*, where the Quotient consists of several Quantities connected by the Signs  $+$  and  $-$ .

60. *The Nature of Division explained, when the Quotient consists of several Quantities connected by the Signs  $+$  or  $-$ .*

To render this the easier to the Learner, let us resume Example 1, Article 22, where we are to divide  $ab + am$  by  $a$ , which being placed as usual in common Arithmetic, thus,

Now

Now the Number of times  $a$  may be had in  $ab$  is  $b$ , that is,  $b$  is the Quotient of  $ab$  divided by  $a$ ; place  $b$  in the Quotient, multiply it by  $a$ , and place the Product  $ab$  as in common Division, and subtracting it from  $ab + am$  the Dividend, there remains  $am$ ; then find how many times  $a$  will go in  $am$ , and it is  $m$ , that is,  $m$  is the Quotient of  $am$  divided by  $a$ , and because the Signs of the Divisor  $a$ , and Dividend  $am$  are alike, therefore it must be  $+m$ , which being placed in the Quotient, and multiplied by  $a$ , the Product is  $am$ , which placing under  $am$ , and subtracting it from  $am$ , there remains 0.

$$a) ab + am (b + m$$

$$\begin{array}{r} ab \\ \hline + am \\ + am \\ \hline 0 \end{array}$$

Hence the Quotient is  $b + m$ .

To divide  $xx + xm + xab$  by  $x$ .

$$\begin{array}{r} x) xx + xm + xab (x + m + ab \\ \underline{xx} \\ xm + xab \\ \underline{xm} \\ xab \\ \underline{xab} \\ 0 \end{array}$$

Here dividing  $xx$  by  $x$ , the Quotient is  $x$ , which placed in the Quotient, and multiplied by the Divisor  $x$ , and placing the Product  $xx$  under the Dividend, from which subtracting it, there remains  $xm + xab$ .

Then dividing  $xm$  by  $x$ , the Quotient is  $m$ , or  $+m$ , for the Signs of  $xm$  and  $x$  are alike, put  $+m$  in the Quotient, by which multiply the Divisor  $x$ , and put the Product  $xm$  under  $xm + xab$ , and subtracting, there remains  $xab$ .

Then dividing  $xab$  by  $x$ , the Quotient is  $ab$ , or  $+ab$ , for the Signs of  $xab$  and  $x$  are alike, put  $+ab$  in the Quotient, by which multiply the Divisor  $x$ , and put the Product  $xab$ , under  $xab$ , and subtracting, there remains 0, hence the Quotient is  $x + m + ab$ .

To

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To divide  $xx + 2xa + aa$  by  $x + a$ .

$$\begin{array}{r}
 x+a)xx+2xa+aa(x+a \\
 \underline{xx+xa} \phantom{+aa} \\
 xa+aa \\
 \underline{xa+aa} \\
 0
 \end{array}$$

Dividing  $xx$  by  $x$ , the Quotient is  $x$ , by which multiplying the Divisor  $x + a$ , the Product is  $xx + xa$ , which placed under the Dividend, and subtracted, there remains  $xa + aa$ .

Then dividing  $xa$  by  $x$ , the Quotient is  $a$ , or  $+a$ , for the Signs of  $xa$  and  $a$  are alike, put  $+a$  in the Quotient, multiplying it by the Divisor  $x + a$ , the Product is  $xa + aa$ , which put under the Remainder  $xa + aa$ , and subtracting, there remains 0, hence the Quotient is  $x + a$ .

To divide  $aa - bb$  by  $a + b$ .

$$\begin{array}{r}
 a+b)aa-bb(a-b \\
 \underline{aa+ab} \\
 -ab-bb \\
 \underline{-ab-bb} \\
 0
 \end{array}$$

Dividing  $aa$  by  $a$ , the Quotient is  $a$ , and multiplying the Divisor by  $a$ , gives  $aa + ab$ , this subtracted from the Dividend leaves  $-ab - bb$ ; for here the Quantity  $ab$ , which is to be subtracted, is, by the Rule for Subtraction, to have its Sign changed, and then added, hence  $+ab$  becomes in the Remainder  $-ab$ .

Then dividing  $-ab$  by  $a$ , the Quotient is  $-b$ , for the Signs of  $ab$  and  $a$  are now unlike; multiplying the Divisor  $a + b$ , by  $-b$ , and subtracting the Product  $-ab - bb$ , from the Remainder  $-ab - bb$ , there remains 0, hence the Quotient is  $a - b$ .

To divide  $aaa - 3aax + 3axx - xxx$  by  $a - x$ .

$$\begin{array}{r}
 a-x)aaa-3aax+3axx-xxx(aa-2ax+xx \\
 \underline{aaa-aa} \phantom{x} \\
 -2aax+3axx-xxx \\
 \underline{-2aax+2ax} \phantom{x} \\
 axx-xxx \\
 \underline{axx-xxx} \\
 0
 \end{array}$$



*The Manner of finding the two Roots, or Values, of the unknown Quantity in Quadratic Equations.*

61. Now to find the other Value of  $a$ , in the *Ambiguous Quadratic Equation*, Question 68.

Take the Work at the Step immediately before you begin to *complete the Square*, which is at the ninth Step, where the Equation is

Make this Equation equal to *nothing*, }  $ee - se = -p$   
by transposing  $p$  }  $ee - se + p = 0$

Then put it in Numbers, and it is  $ee - 45e + 500 = 0$

By the Work we found  $e = 25$

Make this Equation equal to *nothing*, by transposing the 25, thus, }  $e - 25 = 0$

Then divide  $ee - 45e + 500$  by  $e - 25$ , thus,

$$\begin{array}{r} e - 25 \overline{) ee - 45e + 500} \quad (e - 20 \\ \underline{ee - 25e} \phantom{+ 500} \\ - 20e + 500 \\ \underline{- 20e + 500} \\ 0 \end{array}$$

Hence the Quotient is  $e - 20$ , but as the Dividend is *nothing*, for  $ee - 45e + 500 = 0$  as above; and as the Divisor  $e - 25$  is *nothing*, for  $e - 25 = 0$  as above, it follows that the Quotient must be *nothing*, or equal to *nothing*, that is,  $e - 20 = 0$ ; then transposing 20, we have  $e = 20$ , which is the other Value of  $e$ , in this Quadratic *Ambiguous Equation*; therefore, I say the *youngest* Brother was but 20 Years of Age.

And upon this Value of  $e$ , if we take the third Step of the Work to the Question, that is,  $a = s - e$ , we shall find  $a = 25$ , whence the *eldest* Brother was 25 Years of Age, and these are the true Ages of the two Brothers; for their Ages answer the Conditions of the Question, and it is a possible Case, whereas though the other Numbers answered the Conditions of the Question, yet it was impossible for the *youngest* Brother to be 25, when the *eldest* was but 20 Years old.

From the Work of the Question we found	-	$e = 25$
But now we have found	-	$e = 20$
The Sum of these two Values of $e$ , is	-	<u>45</u>

But observing where we put this Quadratic Equation into Numbers, and made it equal to *nothing*, we shall find the Co-efficient of the first Power of  $e$  to be  $-45$ , but the Sum of the two Values of  $e$  is  $+45$ , as above, and concerning these Quadratic Equations, *Algebraists* give us this

## S C H O L I U M.

62. That in Quadratic Equations the Sum of both the Roots, or Values, of the unknown Quantity, is equal to the Co-efficient of the lowest Power of the unknown Quantity, at the Step immediately preceding the completing the Square, but will have the contrary Sign; that is, if the Co-efficient of the lowest Power of the unknown Quantity has the Sign  $+$ , the Sum of both the Roots will be the same as the Co-efficient, but will have the Sign  $-$ .

And if the Co-efficient of the lowest Power of the unknown Quantity has the Sign  $-$ , then the Sum of both the Roots, or Values, will be the same as the Co-efficient, but will have the Sign  $+$ .

Therefore having found any one Root, the other is easily found.

63. To find the other Value of the unknown Quantity in the first Form of Quadratic Equations, or where the Co-efficient of the lowest Power of the unknown Quantity has the Sign  $+$ , it is done by adding the Value of the unknown Quantity found from the Operation, to the Co-efficient of its lowest Power, and to their Sum prefix the Sign  $-$ .

Thus at Question 60, Step 6, the Co-efficient of $e$ , }	10
is $b$ , or	
To which adding the Value of $e$ , as found by that	
Operation,	5
The Sum is	<u>15</u>

And

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And prefixing to this 15 the Sign —, and this is the other Value of  $e$ , that is,  $e = -15$ , which is an *imaginary* Value of  $e$ , it being absurd for a *positive* Quantity to be equal to a *negative* one.

However, we shall find this *imaginary* Value of  $e$ , if we proceed by Division according to the Directions at Art. 61.

For the sixth Step, Question 60, is that which immediately precedes the completing the Square, where the Equation is

$$ee + be = m$$

Which is in Numbers  $ee + 10e = 75$

Transpose 75, to make the Equation }  $ee + 10e - 75 = 0$   
equal to *nothing*

By the Work we found  $e = 5$

Transpose 5, to make this Equation equal to *nothing*  $e - 5 = 0$

Then dividing  $ee + 10e - 75$  by  $e - 5$ .

$$\begin{array}{r} e-5 \overline{) ee + 10e - 75} \quad (e + 15 \\ \underline{ee - 5e} \phantom{- 75} \\ 15e - 75 \\ \underline{15e - 75} \\ 0 \end{array}$$

Hence the Quotient is  $e + 15$ , but as the Dividend is equal to *nothing*, for  $ee + 10e - 75 = 0$ , and as the Divisor  $e - 5$  is equal to *nothing*, for  $e - 5 = 0$ , as above, consequently the Quotient must be equal to *nothing*, that is,  $e + 15 = 0$ , by transposing the 15, we have  $e = -15$ , as before.

For another Example of this Kind take Question 61, where the seventh Step is that which immediately precedes the completing the Square, the Equation being  $ee + me = x - b$ , which being put in Numbers is  $ee + 6e = 112$

By transposing 112, to make the Equation }  $ee + 6e - 112 = 0$   
equal to *nothing*, we have

By the Work it was found  $e = 8$

Transposing 8, to make the Equation equal to }  $e - 8 = 0$   
*nothing*, we have

And dividing to find the other Root of  $e$ , as before,

$$e - 8$$

$$\begin{array}{r}
 e - 8) ee + 6e - 112 (e + 14 \\
 \underline{ee - 8e} \\
 14e - 112 \\
 \underline{14e - 112} \\
 0
 \end{array}$$

Hence the Quotient is  $e + 14$ , which for the same Reason as before, it is  $e + 14 = 0$ , hence  $e = -14$ , for the other Value of  $e$ .

And this Value of  $e$  will be found by the Rule Art. 62.

Thus at Question 61, Step 7, the Co-efficient of $e$	}	6
is $m$ , or	-	-
To which adding the Value of $e$ , found at the	}	8
Operation	-	-
The Sum is	-	14

Then by the Rule prefixing the Sign  $-$  to 14, we have  $-14$  for the other, or *imaginary* Value of  $e$ , the same as before.

But if we add these two Values of  $e$  together, we shall find their Sum answer to the *Scholium*, Art. 62.

The first Value of $e$ is	-	-	-	8
The second Value of $e$ is	-	-	-	-14
				-6

Hence their Sum is the same with the Co-efficient of  $e$ , but has the contrary Sign.

If the Reader has a Mind to prosecute this Speculation, he may try Question 62, Step 9. Question 63, Step 9. Question 64, Step 12, or 13, which are Equations of this first Form, as well as some that follow them.

*To find the other Value or Root of the unknown Quantity in the Second Form of Quadratic Equations.*

64. The second Form of Quadratic Equations, is when the Co efficient of the lowest Power of the unknown Quantity has the Sign  $-$ ; in this Case *subtract the Co-efficient of the lowest Power, supposing it affirmative, of the unknown Quantity in the given Equation, at the Step immediately preceding the completing*



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the Square, from the Value of the unknown Quantity found by the Work, to the Remainder prefix the Sign —, and it will be the other Value of the unknown Quantity. Or place down the Co-efficient with its Sign —, to which add the Value of the unknown Quantity found by the Work, and to this Sum prefix the Sign —, and it will be the other Value, or Root of the unknown Quantity.

An Equation of this second Form is Step 12, Question 65, where we have  $a a - z a = b d - b m x$ .

Here the Co-efficient of  $a$ , is  $-z$ , or  $-28$

And the Value of  $a$  found in that Equation is  $+60$

The Sum is 32, but to it prefix the Sign —, and }  
it is — 32, the other Value of  $a$ , which is *imaginary*, } — 32  
as it has the Sign —.

And if we proceed by Division according to the Directions at Art. 61. we shall find this *imaginary* Value of  $a$ .

Thus if we take the twelfth Step of Question 65, which immediately precedes completing the Square, we have this Equation  $a a - z a = b d - b m x$

Which being put in Numbers is  $a a - 28 a = 1920$

Transposing 1920 to make the }  
Equation equal to *nothing*  $a a - 28 a - 1920 = 0$

By the Work it was found  $a = 60$

Transpose 60 to make the Equation equal to }  
*nothing*  $a - 60 = 0$

And dividing to find the other Root of  $a$ , as before,

$$\begin{array}{r} a - 60) a a - 28 a - 1920 (a + 32 \\ \underline{a a - 60 a} \\ 32 a - 1920 \\ \underline{32 a - 1920} \\ 0 \end{array}$$

Hence the Quotient is  $a + 32$ , which because the Dividend and Divisor are each equal to *nothing*, consequently the Quotient must be equal to *nothing*, hence  $a + 32 = 0$

By transposing 32, we have  $a = -32$ , the same *imaginary* Value of  $a$ , as before.

And

And if we add these two Values of  $a$  together, we shall find their Sum agree with the *Scholium*, Art. 62.

The Value of  $a$  found by the Operation, Question 65, is 60

The Value of  $a$  now found is - - - - - 32

Their Sum is 28, or  $+ 28$ , the same Number as the }  
Co-efficient of  $a$ , but with a contrary Sign - } 28

Another Equation of this second *Form* is Question 67,  
Step 13, where the Equation is  $aa - pa = \frac{zd - rm}{zx}$

Which being put into Numbers is -  $aa - 1.8a = 82$

Transposing 82, to make the Equa- }  
tion equal to *nothing* - - - }  $aa - 1.8a - 82 = 0$

By the Work it was found - - -  $a = 10$

Transposing 10 to make the Equation equal to }  
*nothing* - - - - - }  $a - 10 = 0$

And dividing to find the other Value of  $a$ , as before,

$$\begin{array}{r}
 a - 10) aa - 1.8a - 82 \quad (a + 8.2 \\
 \underline{aa - 10a} \\
 8.2a - 82 \\
 \underline{8.2a - 82} \\
 0
 \end{array}$$

Hence the Quotient is  $a + 8.2$  which must be equal to *nothing*, for the Dividend and Divisor are each equal to *nothing*: but if  $a + 8.2 = 0$ ,

By transposing 8.2 we have  $a = - 8.2$  which is the other Value of  $a$ , and it is *imaginary*, because it has the Sign  $-$ .

The same *imaginary* Value of  $a$  may be found by Art. 64, thus,

The Co-efficient of  $a$ , is - - - - - 1.8

The Value of  $a$  found by the Question, is - - - 10.

The Sum is - - - - - 8.2

Now to this 8.2 prefix the Sign  $-$ , and we have  $- 8.2$  for the *imaginary* Value of  $a$ , the same as before.

And

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And if these two Values of  $a$  are added together, their Sum will agree with the *Scholium*, Art. 62.

The first Value of $a$ , is	-	-	10.
The second Value of $a$ , is	-	-	— 8.2
Sum	-	-	— 1.8
But the Co-efficient of $a$ , is — 1.8			

## 65. But in the *ambiguous*, or third Form of Quadratic Equations,

*If the Value of the unknown Quantity found by the Operation, is subtracted from the Co-efficient of its lowest Power, at the Step immediately before the Square is completed, the Co-efficient being supposed affirmative, the Remainder is its other Value.*

At Question 68, Step 9, the Co-efficient of $e$ is $s$ , or	45
The Value of $e$ , found by the Operation, is	— 25
The Remainder is the other Value of $e$	— 20

And it is this second Value of  $e$  that is the true Answer to the Question, as was observed Page 187; and here the Learner may again observe, that both the Values in this Case are *affirmative*, which makes this be called the *ambiguous* Case, but in the other two preceding Cases, or in the four former Examples, the other Value of the unknown Quantity was *negative*, which is only an *imaginary* Value, it being impossible for an *affirmative*, or *positive* Quantity, which the Question requires, to be a *negative*, or equal to a *negative* Quantity.

But we may find the other Value of  $e$ , in this *ambiguous* Case, by Division, as in the former Instances, thus,

The Equation, Question 68, Step 9, }  
immediately before the Square was com- }  $ee - se = -p$   
pleted, is } - - -

Which being put in Numbers, is -  $ee - 45e = -500$

Transposing 500, to make the }  
Equation equal to *nothing* }  $ee - 45e + 500 = 0$

By the Work it was found - - -  $e = 25$

Transposing 25, to make the Equation equal }  
to *nothing* - - - }  $e - 25 = 0$

And dividing to find the other Value, or Root of  $e$ , as before,

$$\begin{array}{r}
 e - 25 \quad ee - 45e + 500 \quad (e - 20 \\
 \underline{ee - 25e} \\
 - 20e + 500 \\
 \underline{- 20e + 500} \\
 0
 \end{array}$$

Hence the Quotient is  $e - 20$ , which must be equal to *nothing*, for the Reason in the former Cases, but, if  $e - 20 = 0$

Transposing 20, we have  $e = 20$  the other Value of  $e$ , the same as before.

And in this *ambiguous* Case, if we add the two Values of  $e$  together, we shall find them agree with what is said at the *Scholium*, Art. 62.

The first Value of $e$ , is	-	-	25
The second Value of $e$ , is	-	-	20
			45

But the Co-efficient of  $e$ , is  $- 45$ .

*The Manner of expressing the two Roots of an ambiguous Quadratic Equation explained.*

66. Now to explain the usual Manner in which *Algebraists* express the Value of the unknown Quantity, in the *ambiguous* Quadratic Equation; let us resume the Solution of Question 68, at the eighth Step, where there is this

Equation	1	$ee = se - p$
$1 - se$	2	$ee - se = -p$
$2 \text{ c } \square$	3	$ee - se + \frac{ss}{4} = \frac{ss}{4} - p$
$3 \text{ w } 2$	4	$e - \frac{s}{2} = \sqrt{\frac{ss}{4} - p}$
$4 + \frac{s}{2}$	5	$e = \frac{s}{2} \pm \sqrt{\frac{ss}{4} - p}$

That

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That is, prefix both the Signs + and —, to the Quantity under the radical Sign, for that being added to  $\frac{s}{2}$ , or the rational Quantities on that Side of the Equation, gives one of the Values of  $e$ , but if it is subtracted from  $\frac{s}{2}$ , or the rational Quantities on that Side of the Equation, then it gives the other Value of  $e$ , thus,

$$\begin{array}{r}
 s = 45 \\
 s = 45 \\
 \hline
 225 \\
 180 \\
 \hline
 4) 2025 = ss \\
 506.25 = \frac{ss}{4} \\
 - p = 500. \\
 6.25 = \frac{ss}{4} - p \quad (2.5 = \sqrt{\frac{ss}{4} - p} \\
 4 \\
 45) 225 \\
 225 \\
 \hline
 0
 \end{array}$$

Then to find the two Values or Roots of  $e$ .

$$\begin{array}{r}
 \frac{s}{2} = 22.5 \\
 + \sqrt{\frac{ss}{4} - p} = 2.5 \\
 \hline
 25. = \text{one of the Values of } e.
 \end{array}$$

$$\begin{array}{r}
 \frac{s}{2} = 22.5 \\
 - \sqrt{\frac{ss}{4} - p} = -2.5 \\
 \hline
 20.
 \end{array}$$

20. the other Value of  $e$ , which two Values of  $e$  are the same as was found at Art. 61.

And this is the common Method in which *Algebraists* set down, or express the Value of the unknown Quantity, in the *ambiguous* Quadratic Equation.

The Reason of Quadratic Equations having two different Values of the same unknown Quantity, is because the same Quadratic Equation can be formed from two different Suppositions, or Values of the unknown Quantity, or supposing the same unknown Quantity to be equal to two different Numbers.

For let us resume the Equation  $ee - se = -p$ , or  $ee - 45e = -500$ , in this *ambiguous* Equation we found the first Value of  $e$  to be 25, by making  $e$  equal to 25, we have

Multiplying the first Equation by $-45$ , the Co-efficient of $e$ , in the given Equation	$\left. \begin{array}{l} 1 \otimes 2 \\ 2 + 3 \end{array} \right\}$	1	$e = 25$
		2	$ee = 625$
		3	$-45e = -1125$
		4	$ee - 45e = -500$ , the same with the given Quadratic Equation.

And if we take the other Value of  $e$ , viz. 20, we can form the given Equation, for

Let $1 \otimes 2$ $1 \times -45$ the Co- efficient of $e$ , in the given Equation	$\left. \begin{array}{l} 1 \otimes 2 \\ 2 + 3 \end{array} \right\}$	1	$e = 20$
		2	$ee = 400$
		3	$-45e = -900$
		4	$ee - 45e = -500$ , the same with the given Quadratic Equation.

Likewise if we take the first *Form* of Quadratic Equations, viz.  $ee + be = m$ , or  $ee + 10e = 75$ , see Question 60, Step 6. Now the two Values of  $e$  in this Equation we found to be 5, and  $-15$ , and from either of these Values of  $e$ , we can form the given Quadratic Equation,

Suppose $1 \otimes 2$	1	$e = 5$
	2	$ee = 25$

$$\begin{array}{l|l|l}
 1 \times \overline{10} \text{ the Co-eff-} & & \\
 \text{icient of } e, \text{ in the given} & & \\
 \text{Equation} & \left. \begin{array}{l} 3 \\ 2 + 3 \end{array} \right\} & \begin{array}{l} 10e = 50 \\ ee + 10e = 75, \text{ the same with} \\ \text{the given Quadratic Equation.} \end{array}
 \end{array}$$

$$\begin{array}{l|l|l}
 \text{Again, suppose} & 1 & e = -15 \\
 1 \oplus 2 & 2 & ee = 225, \text{ for } -15 \times -15 \\
 & & = +225, \text{ the Signs being} \\
 & & \text{alike.}
 \end{array}$$

$$\begin{array}{l|l|l}
 1 \times \overline{10} \text{ as above} & 3 & 10e = -150 \\
 2 + 3 & 4 & ee + 10e = 75, \text{ the same E-} \\
 & & \text{quation as before.}
 \end{array}$$

And if we take the second *Form* of Quadratic Equations, viz.  $aa - za = bd - bmx$ , or  $aa - 28a = 1920$ , see Question 65, Step 12. The two Values of  $a$  in this Equation we found to be 60, and  $-32$ , from either of which we can form the given Equation, for

$$\begin{array}{l|l|l}
 \text{Suppose} & 1 & a = 60 \\
 1 \oplus 2 & 2 & aa = 3600 \\
 1 \times \overline{28} \text{ the Co-} & & \\
 \text{efficient of } a, \text{ in the} & \left. \begin{array}{l} 3 \\ 2 + 3 \end{array} \right\} & \begin{array}{l} -28a = -1680 \\ aa - 28a = 1920, \text{ the same} \\ \text{with the given Equation.} \end{array}
 \end{array}$$

$$\begin{array}{l|l|l}
 \text{Again, if} & 1 & a = -32 \\
 1 \oplus 2 & 2 & aa = 1024, \text{ for } -32 \times -32 \\
 & & = +1024. \\
 1 \times \overline{28} \text{ the Co-} & & \\
 \text{efficient of } a, \text{ as above} & \left. \begin{array}{l} 3 \\ 2 + 3 \end{array} \right\} & \begin{array}{l} -28a = 896, \text{ for } -28 \times -32 \\ = +896. \\ aa - 28a = 1920, \text{ the same} \\ \text{with the given Equation.} \end{array}
 \end{array}$$

From this the Learner may observe, that making the unknown Quantity equal to either of its Values, and raising this Equation to the Square, and adding it to the former Equation, after it has been multiplied by the Co-efficient of the lowest Power of the unknown Quantity in the Quadratic Equation, this Sum will be the given Quadratic Equation.

Question 69. Two Men, A and B, discoursing of their Shillings, A, who had the greatest Number, said, if my Number of Shillings is divided by your's, and this Quotient is added to your Number of Shillings, the Sum will be 15:

But if the Sum of both our Shillings is multiplied by 4, and this Product divided by 10, the Quotient will be 22. How many Shillings had each Person?

Let  $a$  = the Number of Shillings A had, or the greatest Number,  $e$  = the Number of Shillings B had, or lesser Number,  $s = 15$ ,  $m = 4$ ,  $n = 10$ ,  $d = 22$ .

Then	1	$\frac{a}{e} + e = s$	}	By the Question.
And	2	$\frac{ma + me}{n} = d$		
$1 \times e$	3	$a + ee = se$		
$3 - ee$	4	$a = se - ee$		
$2 \times n$	5	$ma + me = dn$		
$5 - me$	6	$ma = dn - me$		
$6 \div m$	7	$a = \frac{dn - me}{m}$		
$4 \cdot 7$	8	$\frac{dn - me}{m} = se - ee$		
$8 \times m$	9	$dn - me = mse - mee$		
$9 + mee$	10	$mee + dn - me = mse$		
$10 - dn$	11	$mee - me = mse - dn$		
$11 - mse$	12	$mee - me - mse = -dn$		

Here the Equation appears to be *Quadratic*, and of the *ambiguous* Kind; because  $dn$ , the known Side of the Equation, has the *negative* Sign. Then by Art. 58, dividing by  $m$ , the Coefficient of  $ee$ ,

$$12 \div m \quad \left| \quad 13 \quad \right| \quad ee - e - se = -\frac{dn}{m} \quad \text{For } m \text{ be-}$$

ing in every Term on one Side of the Equation, dividing that Part of the Equation by  $m$ , is only to cast away  $m$ , out of every Term of that Side of the Equation, and to divide the other Side



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Side of the Equation is only to place  $m$  as a Denominator to it. The Equation being now prepared for *completing the Square*, and the first Power of  $e$  being in two Terms, *viz.*  $-e - se$ , whose Co-efficients are  $-1$ , and  $-s$ ,

Therefore		Substitute $-z = -1 - s$ , then by
Substitution		14 $ee - ze = -\frac{dn}{m}$
14 c □		15 $ee - ze + \frac{zz}{4} = \frac{zz}{4} - \frac{dn}{m}$
15 w 2		16 $e - \frac{z}{2} = \sqrt{\frac{zz}{4} - \frac{dn}{m}}$
16 + $\frac{z}{2}$	17	$e = \frac{z}{2} \pm \sqrt{\frac{zz}{4} - \frac{dn}{m}} = 8 \pm 3,$ (that is, $e$ is either 5, or 11.)

And if  $e$  is 5, we shall find  $a = 50$ , by the fourth Step, which two Numbers of Shillings answer the Conditions of the Question; or, if we suppose  $e = 11$ , then by the fourth Step we shall find  $a = 44$ , which two Numbers will likewise answer the Conditions of the Question: But sometimes one of the Numbers, or Roots, of these ambiguous Equations, will not answer all the Conditions of the Question, as at Question 74, and then the other Root must be found.

Question 10. Two Merchants, A and B, had gained in Trade, but A, who gained the most, found, that if the Square of the Pounds he gained was multiplied by 2, and the Product added to 8 times the Pounds B gained, if this Sum was divided by 4, the Quotient was 816 Pounds:

But if 3 times the Pounds A gained, was added to 10 times the Pounds B gained, and this Sum divided by 40, the Quotient was 5 Pounds. How many Pounds had each Man gained?

Put  $a =$  the Pounds gained by A,  $e =$  the Pounds gained by B,  $x = 2$ ,  $m = 8$ ,  $p = 4$ ,  $d = 816$ ,  $b = 3$ ,  $z = 10$ ,  $r = 40$ ,  $n = 5$ .

	1	$\frac{x a a + m e}{p} = d$	By the Question.
	2	$\frac{b a + z e}{r} = n$	
$1 \times p$	3	$x a a + m e = p d$	
$3 - x a a$	4	$m e = p d - x a a$	
$4 \div m$	5	$e = \frac{p d - x a a}{m}$	
$2 \times r$	6	$b a + z e = r n$	
$6 - b a$	7	$z e = r n - b a$	
$6 \div z$	8	$e = \frac{r n - b a}{z}$	
$5 \cdot 8$	9	$\frac{r n - b a}{z} = \frac{p d - x a a}{m}$	
$9 \times z$	10	$r n - b a = \frac{z p d - z x a a}{m}$	
$10 \times m$	11	$m r n - m b a = z p d - z x a a$	

Transpose  $z x a a$ , that the highest Power of the unknown Quantity may have the Sign +.

$11 + z x a a$	12	$z x a a + m r n - m b a = z p d$
$12 - m r n$	13	$z x a a - m b a = z p d - m r n$

The Equation now appears to be Quadratic, but to know if it is *ambiguous*, find which Quantity is greatest  $z p d$ , or  $m r n$ , but  $z p d$  is 32640, and  $m r n$  is only 1600, hence  $z p d - m r n = 32640 - 1600 = 31040$ , which being an *affirmative* Number, the Equation is not ambiguous, by Art. 59. But because the Square of the unknown Quantity has a Co-efficient, therefore, by Art. 58,

$13 \div z x$	14	$a a - \frac{m b a}{z x} = \frac{z p d - m r n}{z x}$
		Substitute $-s = -\frac{m b}{z x}$ then by
Substitution	15	$a a - s a = \frac{z p d - m r n}{z x}$
$15 \text{ c } \square$	16	$a a - s a + \frac{s s}{4} = \frac{z p d - m r n}{z x} + \frac{s s}{4}$

16 $\times$ 2	17	$a - \frac{s}{2} = \sqrt{\frac{z p d - m r n}{z x}} + \frac{s s}{4}$
$17 + \frac{s}{2}$	18	$a = \sqrt{\frac{z p d - m r n}{z x}} + \frac{s s}{4} : + \frac{s}{2}$ $(= 40, \text{ the Pounds gained by A.})$
By the eighth Step	19	$e = \frac{r n - b a}{z} = 8, \text{ the Pounds gained by B.}$

Question 71. *A Father, by his Will, left his two Sons, A and B, such a Portion, whereof A had the greatest Fortune; that if the Square of the Number of Pounds he was to have, be multiplied by 2, and to this Product there is added the Number of Pounds B was to have multiplied by 35, the Sum was 6400 Pounds:*

*But if the Number of Pounds A was to have, be multiplied by 20, and this Product added to the Number of Pounds B was to have multiplied by 15, the Sum was 1600 Pounds. To find the Fortune of each?*

Let  $a$  = the Fortune of A,  $e$  = the Fortune of B,  $x = 2$ ,  $m = 35$ ,  $d = 6400$ ,  $b = 20$ ,  $z = 15$ ,  $r = 1600$ .

	1	$x a a + m e = d$	} By the Question.
	2	$b a + z e = r$	
1 — $x a a$	3	$m e = d - x a a$	
$3 \div m$	4	$e = \frac{d - x a a}{m}$	
2 — $b a$	5	$z e = r - b a$	
$5 \div z$	6	$e = \frac{r - b a}{z}$	
4 . 6	7	$\frac{r - b a}{z} = \frac{d - x a a}{m}$	
$7 \times z$	8	$r - b a = \frac{z d - z x a a}{m}$	
$8 \times m$	9	$m r - m b a = z d - z x a a$	

Transpose  $z x a a$ , that the highest Power of  $a$  may be affirmative.

9 + $z x a a$	10	$z x a a + m r - m b a = z d$
10 — $m r$	11	$z x a a - m b a = z d - m r$
		D d

The

The Equation now appears to be *Quadratic*, and to know if it is *ambiguous*, find what  $zd$  and  $mr$  are in Numbers. But  $zd - mr = 96000 - 56000 = 40000$ , a *positive* Quantity, whence the Equation is not *ambiguous* by Art. 59. And because the Square of the unknown Quantity has a Co-efficient, therefore by Art. 58.

11 ÷ zx	12	$aa - \frac{mba}{zx} = \frac{zd - mr}{zx}$
		Substitute $-s = -\frac{mb}{zx}$ then by
Substitution	13	$aa - sa = \frac{zd - mr}{zx}$
13 c □	14	$aa - sa + \frac{ss}{4} = \frac{zd - mr}{zx} + \frac{ss}{4}$
14 w 2	15	$a - \frac{s}{2} = \sqrt{\frac{zd - mr}{zx} + \frac{ss}{4}}$
15 + $\frac{s}{2}$	16	$a = \sqrt{\frac{zd - mr}{zx} + \frac{ss}{4}} + \frac{s}{2}$

$= 49.9999$ , &c. because of the Imperfection of the Decimal Fraction; the true Number being 50, from which by

The sixth Step  $\left| 17 \right| e = \frac{r - ba}{z} = 40.$

Question 72. *What are those two Numbers, the Quotient of the greater divided by 5, and added to the lesser, the Sum may be 12:*

*But the Product of the two Numbers divided by 4, the Quotient is 40?*

Put  $a$  = the greater Number,  $e$  = the lesser Number,  $m = 5$ ,  $p = 12$ ,  $d = 4$ ,  $x = 40$ .

	1	$\frac{a}{m} + e = p$	} By the Question.
	2	$\frac{ae}{d} = x$	
1 × m	3	$a + me = mp$	
3 - me	4	$a = mp - me$	
2 × d	5	$ae = dx$	

5 ÷ e	6	$a = \frac{dx}{e}$
4 · 6	7	$\frac{dx}{e} = mp - me$
7 × e	8	$dx = mpe - mee$
8 + mee	9	$mee + dx = mpe$
9 - dx	10	$mee = mpe - dx$
10 - mpe	11	$mee - mpe = -dx$

Here the Equation not only appears *Quadratic*, but *Ambiguous*, for  $dx$  the known Side of the Equation is *negative*. Now by Art. 58.

11 ÷ m	12	$ee - pe = -\frac{dx}{m}$
12 c □	13	$ee - pe + \frac{pp}{4} = \frac{pp}{4} - \frac{dx}{m}$
13 w 2	14	$e - \frac{p}{2} = \sqrt{\frac{pp}{4} - \frac{dx}{m}}$
14 + $\frac{p}{2}$	15	$e = \frac{p}{2} \pm \sqrt{\frac{pp}{4} - \frac{dx}{m}} = 6 \pm 2,$ that is, $e$ is either 8, or 4: But if $e = 8$ , then by

The sixth Step	16	$a = \frac{dx}{e} = 20.$ Or if $e = 4$ , then
----------------	----	---

$a = 40$ , either of which answers the Question.

**Question 73.** Two young Gentlemen having been at the Gaming-Tables, and being asked by their Friend what they lost, which being ashamed to own, A said, if the Number of Pounds I lost is divided by 4, and this added to the Number of Pounds B lost divided by 2, the Sum is 9 Pounds:

But if the Product of the Number of Pounds we both lost is divided by 10, and extracting the square Root of this Quotient, it will be 4. How much did each Person lose?

Let  $a$  = the Number of Pounds lost by A,  $e$  = the Number of Pounds lost by B,  $b = 4$ ,  $d = 2$ ,  $m = 9$ ,  $p = 10$ : as the Number 4 is in the first Part of the Question, and it being again repeated, there is no Occasion for any new Letter.

1	$\frac{a}{b} + \frac{e}{d} = m$	}	By the Question.
2	$\sqrt{\frac{ae}{p}} = b$		
1 — $\frac{e}{d}$	3	$\frac{a}{b} = m - \frac{e}{d}$	
3 × b	4	$a = bm - \frac{be}{d}$	
2 ⊙ 2	5	$\frac{ae}{p} = b b$	
5 × p	6	$ae = p b b$	
6 ÷ e	7	$a = \frac{p b b}{e}$	
4 · 7	8	$\frac{p b b}{e} = bm - \frac{be}{d}$	
8 × e	9	$p b b = b m e - \frac{b e e}{d}$	
9 × d	10	$d p b b = d b m e - b e e$	
10 ÷ b	11	Dividing by $b$ , by Art. 53. $d p b = d m e - e e$	
11 + e e	12	To have the highest Power of $e$ affirmative, transpose $ee$ . $ee + d p b = d m e$	
12 — d p b	13	$ee = d m e - d p b$	
13 — d m e	14	$ee - d m e = -d p b$	
14 c □	15	Here the Equation appears <i>quadratic</i> , and <i>ambiguous</i> . $ee - d m e + \frac{d d m m}{4} = \frac{d d m m}{4} - d p b$	
15 w 2	16	$e - \frac{d m}{2} = \sqrt{\frac{d d m m}{4} - d p b}$	
16 + $\frac{d m}{2}$	17	$e = \frac{d m}{2} \pm \sqrt{\frac{d d m m}{4} - d p b} = 9 \pm 1,$	

that is,  $e$  is either 8, or 10, whence by the fourth, or seventh Steps, we shall find  $a = 20$ , or 16.

Question 74. In the right-angled Triangle ABC, there is given the Hypotenuse AC = 10, and the Sum of the Base AB and Perpendicular BC = 14. To find the Base AB and Perpendicular BC? See Figure, Page 206.

Let

# Of Quadratic EQUATIONS. 205

Let  $AC = b = 10$ ,  $AB + BC = d = 14$ ,  $AB = a$ ; and because  $AB + BC = d$ , therefore from this subtracting  $AB$ , or  $a$ , we have  $BC = d - a$ .

Having expressed all the Sides of the Figure in Symbols, and there being but one unknown Quantity, we are only to raise one Equation from the Property of the Figure; and the Triangle  $ABC$  being right-angled, we have by 47 *e* 1 the Square of the Hypotenuse  $AC$ , or  $bb$ , equal to the Square of the Base  $AB$ , or  $aa$ , added to the Square of the Perpendicular  $BC$ , or  $dd - 2da + aa$ , that is,

$$\begin{array}{l|l} \text{In Symbols} & 1 \\ 1 - dd & 2 \end{array} \left| \begin{array}{l} bb = dd - 2da + 2aa \\ 2aa - 2da = bb - dd \end{array} \right.$$

Here the Equation appears *quadratic*, and because  $dd$  is greater than  $bb$ , it is likewise *ambiguous*, for  $bb - dd = 100 - 196 = -96$  a *negative* Quantity; but as the Square of the unknown Quantity has a Co-efficient, therefore divide by it by Art. 58.

$$\begin{array}{l|l} 2 \div -2 & 3 \\ 3 \text{ c } \square & 4 \\ 4 \text{ w } 2 & 5 \\ 5 + \frac{d}{2} & 6 \end{array} \left| \begin{array}{l} aa - da = \frac{bb - dd}{2} \\ aa - da + \frac{dd}{4} = \frac{dd}{4} + \frac{bb - dd}{2} \\ a - \frac{d}{2} = \sqrt{\frac{dd}{4} + \frac{bb - dd}{2}} \\ a = \frac{d}{2} \pm \sqrt{\frac{dd}{4} + \frac{bb - dd}{2}} = 7 \pm 1 \end{array} \right.$$

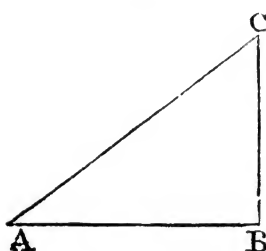
from whence the Base  $AB$  may be either 8, or 6; supposing the Base 8, then because by the Question, the Sum of the Base and Perpendicular is 14, the Perpendicular  $BC$  will be 6; but if we suppose the Base to be 6, then from the same Reasoning the Perpendicular  $BC$  will be 8.

And the Question not limiting which is longest, either the Base  $AB$ , or Perpendicular  $BC$ , we may take either 6, or 8, for the Length of the Base  $AB$ , for either will answer the Conditions of the Question.

But if the Question had said that the Base  $AB$ , is longer than the Perpendicular  $BC$ , then we must take  $a = \frac{d}{2}$

$$+ \sqrt{\frac{dd}{4} + \frac{bb - dd}{2}} = 8, \text{ by which we shall find the Perpendicular}$$

dicular  $BC = 6$ ; for if we take  $a = \frac{d}{2} - \sqrt{\frac{dd}{4} + \frac{bb-dd}{2}}$   
 $= 6$ , then we shall find the Perpendicular  $BC = 8$ , which cannot be, because the Question is supposed to determine the Base  $AB$ , to be longer than the Perpendicular  $BC$ .



Question 75. In the right-angled Triangle  $ABC$ , given the Hypotenuse  $AC = 10$ , the Perpendicular  $BC$ , being shorter than the Base  $AB$ , by subtracting the Perpendicular  $BC$  from the Base  $AB$ , and multiplying the Difference by 20, and dividing this Product by 8, the Quotient is 5. What is the Length of the Base  $AB$ , and Perpendicular  $BC$ ?

Let  $AC = b = 10$ ,  $AB = a$ ,  $BC = e$ ,  $d = 20$ ,  $m = 8$ ,  $z = 5$ .

1	1	$aa + ee = bb$ by the Property of the Figure, as in the last Question.
2	2	$\frac{da - de}{m} = z$ by the Question.
3	3	$aa = bb - ee$
4	4	$a = \sqrt{bb - ee}$
5	5	$da - de = zm$
6	6	$da = zm + de$
7	7	$a = \frac{zm + de}{d}$
8	8	$\frac{zm + de}{d} = \sqrt{bb - ee}$

Squaring both Sides of the Equation, because the unknown Quantity is under the radical Sign.

8 ⊕ 2	9	$\frac{zzmm + 2zmdc + ddee}{dd} = bb - ee$
9 × dd	10	$zzmm + 2zmdc + ddee = ddbb - ddee$
10 + ddee	11	$zzmm + 2zmdc + 2ddee = bbdd$
11 - zzmm	12	$2ddee + 2zmdc = bbdd - zzmm$
2		Dividing



		Dividing by the Co-efficient of $ee$ by Art. 58.
$12 \div 2dd$	13	$ee + \frac{2zmed}{2dd} = \frac{bbdd - zzm m}{2dd}$
That is	14	$ee + \frac{zme}{d} = \frac{bbdd - zzm m}{2dd}$ for
		$\frac{2zmed}{2dd} = \frac{zme}{d}$ , rejecting $2d$ , as in Division.

Hence the Equation is *quadratic*, but it cannot be *ambiguous*, because both the Quantities  $ee + \frac{zme}{d}$  having the Sign  $+$ , the whole Side of the Equation must be *affirmative*, and consequently the other Side of the Equation must be also *affirmative*, otherwise an *affirmative* Quantity would be equal to a *negative* Quantity, which is absurd. Now,

		Substitute $x = \frac{zm}{d} = 2$ , by Art. 57.
		then by
Substitution	15	$ee + xe = \frac{bbdd - zzm m}{2dd}$
$15 \text{ c } \square$	16	$ee + xe + \frac{xx}{4} = \frac{bbdd - zzm m}{2dd}$
		$+ \frac{xx}{4}$
$16 \text{ wz}$	17	$e + \frac{x}{2} = \sqrt{\frac{bbdd - zzm m}{2dd} + \frac{xx}{4}}$
$17 - \frac{x}{2}$	18	$e = \sqrt{\frac{bbdd - zzm m}{2dd} + \frac{xx}{4}} - \frac{x}{2}$
		$= 6 = BC.$
Then by Step 7th	19	$a = \frac{zm + de}{d} = 8 = AB.$

*The same Question done otherwise.*

Let  $AC = b = 10$ ,  $AB = a$ , then by 47 e 1,  $BC = \sqrt{bb - aa}$ ; suppose  $d = 20$ ,  $m = 8$ ,  $z = 5$ , as before.

Now

Now all the Sides of the Triangle being expressed in Symbols, and there being only one unknown Quantity, there is but one Equation required, which may be raised from the Conditions of the Question, and these I shall particularly express to prevent any Difficulty to the Learner.

Now | 1 |  $a$ , is the Base AB, which is longer than the Perpendicular BC, or  $\sqrt{bb - aa}$ , therefore connecting  $\sqrt{bb - aa}$  to  $a$  by the Sign —,

We have | 2 |  $a - \sqrt{bb - aa}$  for the Difference between the Base and Perpendicular, which is to be multiplied by 20, or  $d$ , then

We have | 3 |  $da - d\sqrt{bb - aa}$

But this Product is to be divided by 8, or  $m$ , then

We have | 4 |  $\frac{da - d\sqrt{bb - aa}}{m}$ , and this Quotient (is to be equal to 5, or  $z$ ).

Whence | 5 |  $\frac{da - d\sqrt{bb - aa}}{m} = z$ , by the (Question.

Because the unknown Quantity is divided by  $m$ , therefore by Art. 47.

$5 \times m$  | 6 |  $da - d\sqrt{bb - aa} = zm$

Because the unknown Quantity is multiplied by  $d$ , therefore by Art. 48.

$6 \div d$  | 7 |  $a - \sqrt{bb - aa} = \frac{zm}{d}$

Now transpose the Surd, because it has the Sign —, the highest Power of the unknown Quantity being Part of it.

$7 + \sqrt{bb - aa}$  | 8 |  $a = \frac{zm}{d} + \sqrt{bb - aa}$   
 $8 - \frac{zm}{d}$  | 9 |  $\sqrt{bb - aa} = a - \frac{zm}{d}$

There being no Quantities on the same Side of the Equation with the Surd, raise both Sides of the Equation to the second Power to take away the radical Sign.

9 ⊖ 2	10	$bb - aa = aa - \frac{2zma}{d} + \frac{zzmm}{dd}$
10 + aa	11	$2aa - \frac{2zma}{d} + \frac{zzmm}{dd} = bb$
11 - $\frac{zzmm}{dd}$	12	$2aa - \frac{2zma}{d} = bb - \frac{zzmm}{dd}$
12 - 2	13	$aa - \frac{zma}{d} = \frac{bb}{2} - \frac{zzmm}{2dd}$

Here the Equation is *quadratic*, but because  $\frac{bb}{2} - \frac{zzmm}{2dd} = 50 - 2 = 48$ , a *positive* Quantity, it is not *ambiguous*.  
 Now by Art. 57, substitute  $-x = -\frac{zm}{d} = -2$ .

Then	14	$aa - xa = \frac{bb}{2} - \frac{zzmm}{2dd}$
14 c □	15	$aa - xa + \frac{xx}{4} = \frac{xx}{4} + \frac{bb}{2} - \frac{zzmm}{2dd}$
15 w 2	16	$a - \frac{x}{2} = \sqrt{\frac{xx}{4} + \frac{bb}{2} - \frac{zzmm}{2dd}}$
16 + $\frac{x}{2}$	17	$a = \frac{x}{2} + \sqrt{\frac{xx}{4} + \frac{bb}{2} - \frac{zzmm}{2dd}}$ (= 8 = the Base AB, as before.

Hence in the right-angled Triangle ABC, because we have given the Hypothenufe AC, which is 10, and having now found the Base AB to be 8, therefore the Perpendicular BC =  $\sqrt{100 - 64} = 6$ .

Question 76. Two Merchants, A and B, becoming Bankrupts, owe such Sums of Money, that if from the Number of Pounds A owes, we subtract the Square of the Number of Pounds B owes, there remains 1900 Pounds :

But if the Square of the Number of Pounds B owes, is multiplied by the Number of Pounds A owes, the Product is 81000000 Pounds. To find the Debt of each Merchant ?

E e

Let

Let  $a$  = the Money owed by A,  $e$  = the Money owed by B,  $b = 1900$ ,  $d = 81000000$ .

	1	$a - ee = b$	} By the Question.
	2	$aee = d$	
$1 + ee$	3	$a = b + ee$	
$2 \div ee$	4	$a = \frac{d}{ee}$	
$3 \cdot 4$	5	$ee + b = \frac{d}{ee}$	
$5 \times ee$	6	$eeee + b ee = d$	

Perhaps this Equation may appear new to the young *Analyst*, but by turning to Art. 56. he will find it to be a *Quadratic* Equation, for the unknown Quantity is only in two Terms, and in one of them its Power or Height is *double* its Power or Height in the other, for it is  $eeee$  and  $ee$ , therefore take  $b$  the Co-efficient of  $ee$ , the lowest Power of  $e$  in the present Case; divide it by 2, square the Quotient, and add it to both Sides of the Equation, as before, thus,

$6 c \square$	7	$eeee + b ee + \frac{bb}{4} = d + \frac{bb}{4}$	
		Extracting the square Root as usual,	
$7 w 2$	8	$ee + \frac{b}{2} = \sqrt{d + \frac{bb}{4}}$	
		Transposing $\frac{b}{2}$ because it is a known Quantity,	
$8 - \frac{b}{2}$	9	$ee = \sqrt{d + \frac{bb}{4}} - \frac{b}{2}$	
		Now extracting the square Root to deprefs $e$ to the first Power.	
$9 w 2$	10	$e = \sqrt{\sqrt{d + \frac{bb}{4}} - \frac{b}{2}} = 90 \text{ Pounds,}$	
		(the Money B owed.	

To extract the Square Root of the Quantity  $\sqrt{d + \frac{bb}{4}} - \frac{b}{2}$ , is only to place again the radical Sign before the same Quantity,

Quantity, drawing it over the radical Sign already there, and the other Quantities without that Sign, if there are any; for though these were not included in the first Root, yet as they were afterwards transposed to that Side of the Equation, and the Root is again required to be taken, they will now be included under the radical Sign of this second Extraction.

Because of the two radical Signs, I shall set down the Numerical Work, to make the Operation the plainer.

$$\begin{array}{rcl}
 d & = & 81000000 \\
 bb & = & 902500 \\
 4 & \overline{) 81902500} & = d + \frac{bb}{4} \quad (9050 = \sqrt{d + \frac{bb}{4}} \\
 & 81 & - 950 = -\frac{b}{2} \\
 1805 \overline{) 9025} & & \frac{8100 = \sqrt{d + \frac{bb}{4}} : -\frac{b}{2}}{\text{and the square Root of } 8100} \\
 & 9025 & \text{is } 90 = \sqrt{\sqrt{d + \frac{bb}{4}} : -\frac{b}{2}} \\
 & \underline{0} & \text{Then by the third Step } a = b + ee \\
 & & = 10000 \text{ Pounds, the Money} \\
 & & \text{owed by A.}
 \end{array}$$

The same Question answered by exterminating the unknown Quantity  $e$ .

$$\begin{array}{l|l|l}
 1 - ee & 1 & a - ee = b \\
 3 - b & 2 & a ee = d \\
 2 \div a & 3 & a = b + ee \\
 & 4 & ee = a - b \\
 & 5 & ee = \frac{d}{a}
 \end{array} \quad \left. \vphantom{\begin{array}{l|l|l} 1 - ee \\ 3 - b \\ 2 \div a \end{array}} \right\} \begin{array}{l} \text{By the Question, as} \\ \text{before.} \end{array}$$

Make the fourth and fifth Equations equal to one another, for each is equal to  $ee$ .

$$\begin{array}{l|l|l}
 4 \cdot 5 & 6 & a - b = \frac{d}{a} \\
 6 \times a & 7 & aa - ba = d
 \end{array}$$

E e 2

$$\begin{array}{l|l}
 7 \text{ c } \square & 8 \left| \begin{array}{l} a a - b a + \frac{b b}{4} = d + \frac{b b}{4} \\ a - \frac{b}{2} = \sqrt{d + \frac{b b}{4}} \\ a = \sqrt{d + \frac{b b}{4}} + \frac{b}{2} = 10000 \end{array} \right. \\
 8 \text{ w } 2 & 9 \\
 9 + \frac{b}{2} & 10
 \end{array}$$

(Pounds, as before.)

And by the fourth Step  $e = \sqrt{a - b}$ , or by the fifth Step,  $e = \sqrt{\frac{d}{a}} = 90$  Pounds, as before.

From hence the Learner may observe, there are different Methods of answering the same Question, and that some are more elegant than others, as they give the Answer in more simple or less complicated Terms: And in this Part of the Science he is to exercise himself according to his own Prudence and Judgment, and some Measure in Proportion as he understands and conceives the general and universal Methods by which Questions are answered; it being only my Design to illustrate these by pertinent Examples, with such Solutions as arise in an obvious Manner from the Directions, that the Learner may acquire some general Idea of the Nature and Excellency of Algebra.

*Question 75. Two Running Footmen, A and B, meeting on the Road, found, if the Number of Miles A had run was multiplied by 5, and subtracting from this Product the Square of the Miles run by B, there remained 100:*

*But if the Square of the Miles run by B, was multiplied by the Number of Miles run by A, and the Product multiplied by 2, this Product was 80000. How many Miles had each Person run?*

Let  $a$  = the Number of Miles run by A,  $e$  = the Number of Miles run by B,  $m = 5$ ,  $x = 100$ ,  $d = 2$ ,  $b = 80000$ .

$$\begin{array}{l|l}
 1 + e e & 1 \left| \begin{array}{l} m a - e e = x \\ d a e e = b \end{array} \right. \\
 3 \div m & 2 \left| \begin{array}{l} m a = x + e e \\ a = \frac{x + e e}{m} \end{array} \right. \\
 2 \div d e e & 3 \left| \begin{array}{l} a = \frac{b}{d e e} \end{array} \right.
 \end{array}$$

} By the Question.

$$\begin{array}{l|l|l} 4 \cdot 5 & 6 & \frac{x+ee}{m} = \frac{b}{dee} \\ 6 \times m & 7 & ee+x = \frac{mb}{dee} \\ 7 \times dee & 8 & deeee+xdee=mb \end{array}$$

Here the Equation appears to be of the same Kind with the last, that is quadratic, but not ambiguous. Now by Art. 58.

$$\begin{array}{l|l|l} 9 \div d & 9 & eeee+xee = \frac{mb}{d} \\ & & \text{And completing the Square as in the} \\ & & \text{last Question,} \\ 10 \text{ c } \square & 10 & eeee+xee+\frac{xx}{4} = \frac{mb}{d} + \frac{xx}{4} \\ 11 \text{ w } 2 & 11 & ee+\frac{x}{2} = \sqrt{\frac{mb}{d} + \frac{xx}{4}} \\ 12 - \frac{x}{x} & 12 & ee = \sqrt{\frac{mb}{d} + \frac{xx}{4}} : -\frac{x}{2} \\ 13 \text{ w } 2 & 13 & e = \sqrt{\sqrt{\frac{mb}{d} + \frac{xx}{4}} : -\frac{x}{2}} \end{array}$$

$$2) 400000 = mb$$

$$200000 = \frac{mb}{d}$$

$$2500 = \frac{xx}{4}$$

$$202500 (450 = \sqrt{\frac{mb}{d} + \frac{xx}{4}}$$

$$16 - 50 = -\frac{x}{2}$$

$$85) 425$$

$$\frac{425}{0}$$

$$400 (20 = \sqrt{\sqrt{\frac{mb}{d} + \frac{xx}{4}} : -\frac{x}{2}} = \text{the}$$

(Number of Miles run by B.

Then by the fourth Step  $a = \frac{x+ee}{m} = 100$ , the Number of Miles run by A.

This

This Question, as the last, may be resolved in a more simple Manner, if we exterminate the unknown Quantity  $e$  instead of  $a$ , thus,

		1	$ma - ee = x$	} By the Question, as before
		2	$daee = b$	
$1 + ee$		3	$ma = x + ee$	
$3 - x$		4	$ee = ma - x$	
$2 \div da$		5	$ee = \frac{b}{da}$	
$4 \cdot 5$		6	$ma - x = \frac{b}{da}$	
$6 \times da$		7	$dmaa - dxa = b$	

Dividing by  $dm$  the Co-efficient of  $aa$ , by Art. 58.

$7 \div dm$		8	$aa - \frac{xa}{m} = \frac{b}{dm}$ for $\frac{dxa}{dm} = \frac{xa}{m}$ the $d$ being rejected as in Division.
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Now  $\frac{x}{m}$  the Co-efficient of  $a$  being divided by 2, is  $\frac{x}{2m}$ . For making 2 an improper Fraction by the Rule in common Arithmetic, is  $\frac{2}{1}$ : But by the Rule for Division of Vulgar Fractions in Arithmetic  $\left(\frac{2}{1}\right) \frac{x}{m} \left(\frac{x}{2m}\right)$ . Now squaring  $\frac{x}{2m}$  is  $\frac{xx}{4mm}$ , and adding this to both Sides of the Equation, the Square is completed, by Art. 56.

$8 \epsilon \square$		9	$aa - \frac{xa}{m} + \frac{xx}{4mm} = \frac{b}{dm} + \frac{xx}{4mm}$
$9 \text{ uw } 2$		10	$a - \frac{x}{2m} = \sqrt{\frac{b}{dm} + \frac{xx}{4mm}}$
$10 + \frac{x}{2m}$		11	$a = \sqrt{\frac{b}{dm} + \frac{xx}{4mm}} + \frac{x}{2m} = 100$ (as before.)

Then by the fourth or fifth Step we shall find  $e = 20$ , as before.

Question



# Of Quadratic EQUATIONS. 215

Question 78. It is required to find two such Numbers, that the greater being added to the Square of the lesser, the Sum may be 19 :

But if the greater is multiplied into the Square of the lesser, the Product may be 90.

Let  $a$  = the greater Number,  $e$  = the lesser Number,  $s = 19$ ,  $p = 90$ .

	1	$a + ee = s$	} By the Question.
	2	$aee = p$	
$1 - ee$	3	$a = s - ee$	
$2 \div ee$	4	$a = \frac{p}{ee}$	
$3 \cdot 4$	5	$\frac{p}{ee} = s - ee$	
$5 \times ee$	6	$p = see - eeee$	
		Transpose eeee to make it affirmative.	
$6 + eeee$	7	$eeee + p = see$	
$7 - see$	8	$eeee - see + p = 0$	
$8 - p$	9	$eeee - see = -p$	

Here the Equation not only appears *quadratic*, the Powers of the unknown Quantity  $e$ , being the same as in the two last Questions, but it is likewise *ambiguous*, for that Side of the Equation which is known is *negative*, viz.  $-p$ .

$9 \square$	10	$eeee - see + \frac{ss}{4} = \frac{ss}{4} - p$
$10 \text{ w } 2$	11	$ee - \frac{s}{2} = \sqrt{\frac{ss}{4} - p}$
$11 + \frac{s}{2}$	12	$ee = \frac{s}{2} \pm \sqrt{\frac{ss}{4} - p}$ the Equation
		(being <i>ambiguous</i> , as above.)
$12 \text{ w } 2$	13	$e = \sqrt{\frac{s}{2} \pm \sqrt{\frac{ss}{4} - p}}$

That is, by reason of the *Ambiguity* of the Equation, it may

$$\text{be } e = \sqrt{\frac{s}{2} + \sqrt{\frac{ss}{4} - p}}, \text{ or } e = \sqrt{\frac{s}{2} - \sqrt{\frac{ss}{4} - p}}$$

Let

Let us suppose  $e = \sqrt{\frac{s}{2} + \sqrt{\frac{ss}{4} - p}}$

$$\begin{array}{r} 90.25 = \frac{ss}{4} \\ -90. = -p \\ \hline \sqrt{.25} = .5 = \sqrt{\frac{ss}{4} - p} \end{array}$$

$$\begin{array}{r} 9.5 = \frac{s}{2} \\ \hline 10. = \frac{s}{2} + \sqrt{\frac{ss}{4} - p} \end{array}$$

$$10) 3.162 \text{ nearest} = \sqrt{\frac{s}{2} + \sqrt{\frac{ss}{4} - p}} \text{ the Value (of } e.$$

$$\begin{array}{r} 9 \\ 61 \overline{) 100} \\ \underline{61} \\ 626 \overline{) 3900} \\ \underline{3756} \\ 6322 \overline{) 14400} \\ \underline{12644} \end{array}$$

Then by the third Step  $a = s - ee = 9$ , if we take 10 for the Square of  $e$ , the square Root of 10 being equal to  $e$ .

By trying these Numbers according to the Conditions of the Question, we have

$$\begin{array}{l} a + ee = 19 \\ a ee = 90 \text{ taking 10 for the Square of } e, \text{ as above.} \end{array}$$

But because the Value of  $e$  is a Fraction which does not terminate, and therefore its exact Value cannot be found, let us

try the other Root, *viz.*  $e = \sqrt{\frac{s}{2} - \sqrt{\frac{ss}{4} - p}}$

$$\begin{array}{r} 90.25 = \frac{ss}{4} \\ -90. = -p \\ \hline \sqrt{.25} = .5 = \sqrt{\frac{ss}{4} - p} \end{array}$$

$$\frac{s}{2} = 9.5$$

$$-\sqrt{\frac{ss}{4} - p} = -.5$$

$$\frac{s}{2} - \sqrt{\frac{ss}{4} - p} = 9. \text{ extracting the square Root of 9. we have}$$

$$\sqrt{\frac{s}{2} - \sqrt{\frac{ss}{4} - p}} = 3, \text{ for the other Value of } e \text{ exact.}$$

Then by the third Step  $a = s - ee = 10$ .

And trying these two Numbers by the Conditions of the Question, we have

$$\left. \begin{array}{l} a + ee = 19 \\ aee = 90 \end{array} \right\} \text{ As the Question requires, whence the two Numbers are 10 and 3.}$$

I have been particular in the Arithmetical Work of this Question, that the Learner may see the Method of finding both the Values of the unknown Quantity, in any *ambiguous* quadratic Equation, when the unknown Quantity is to the fourth Power.

But in this Question, if we exterminate  $e$  instead of  $a$ , we shall have a more simple Solution.

1 — a	1	a + ee = s	} By the Question as before.	
	2	aee = p		
	3	<hr style="width: 100%; border: 0.5px solid black;"/> ee = s — a		
	4	ee = $\frac{p}{a}$		
	4 · 3	5	$\frac{p}{a} = s - a$	
5 × a	6	p = sa — aa		
6 + aa	7	aa + p = sa		
7 — p	8	aa = sa — p		
8 — sa	9	aa — sa = — p		

Here the Equation appears *quadratis* and *ambiguous*, as before.

$$9c \square \mid 10 \mid aa - sa + \frac{ss}{4} = \frac{ss}{4} - p$$

$$\begin{array}{l|l} \text{IO} \text{ or } 2 & \text{II} \\ \hline & a - \frac{s}{2} = \sqrt{\frac{ss}{4} - p} \\ \text{II} + \frac{s}{2} & \text{I2} \\ \hline & a = \frac{s}{2} \pm \sqrt{\frac{ss}{4} - p} \end{array}$$

Let us first suppose the Root to be  $\frac{s}{2} - \sqrt{\frac{ss}{4} - p}$

$$\begin{array}{r} 4) 361 = ss \\ \hline 90.25 = \frac{ss}{4} \\ - p = - 90. \\ \hline .25 = \frac{ss}{4} - p, \text{ but the square Root of } .25 \text{ is } .5 \\ \text{whence } .5 = \sqrt{\frac{ss}{4} - p} \end{array}$$

$$\begin{array}{r} \text{Then } \frac{s}{2} = 9.5 \\ - \sqrt{\frac{ss}{4} - p} = - 0.5 \\ \hline \end{array}$$

$9. = a$ , for one of the Roots of the ambiguous Equation, and from this Root, or Value of  $a$ , we shall from the third, or fourth Step, find, that  $e$  is equal to the square Root of 10, as before; but this being a furd Number, whose Root cannot be exactly extracted, therefore find the other

Root, or Value of  $a$ , then we have  $a = \frac{s}{2} + \sqrt{\frac{ss}{4} - p}$ .

$$\begin{array}{r} \frac{s}{2} = 9.5 \\ + \sqrt{\frac{ss}{4} - p} = .5 \\ \hline \end{array}$$

$10. = a$ , the other Root of the ambiguous Equation; then by the third, or fourth Step, we shall find  $e$  to be equal to the square Root of 9, which is 3; and these two Numbers 10, and 3, answer the Conditions of the Question.

It may not be improper in this Place to add, that if the Learner meets with any Questions, where the Answers come  
out

out in Decimal Fractions, he is not from thence to conclude they are not the true Answers, as these are very frequent and common: But if the Equation is *ambiguous*, it will be proper to find the other Root, which may be free from Fractions; and if this Root answers the Conditions of the Question, he has then found the Answer compleat: But if the Question will not admit of such an Answer, he can then only *approach* to the true Answer in continuing his Fractions at Pleasure; but hitherto I have endeavoured to avoid these Circumstances, as they only fatigue the Learner, and perplex his Mind, instead of increasing his Judgment, or advancing his Knowledge in this Science.

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## 66. *The Method of resolving Questions, that contain three Equations, and three unknown Quantities.*

**FIND** the Value of one of the unknown Quantities, in one of the given Equations:

For the same unknown Quantity in the other two Equations, write, or put this Value, which exterminates that unknown Quantity from those two Equations; and reduces the Question to two Equations, and two unknown Quantities, which may be resolved as the foregoing Questions, by Art. 55. that is,

Find the Value of one of these two unknown Quantities, in each of those two Equations, and making these two Equations equal to one another, *exterminates* another unknown Quantity, for this last Equation will have only one unknown Quantity, which being reduced by the Directions already given, will give the Value of that unknown Quantity in Numbers, from which it will be easy to determine the Value of the other two.

To help the Learner in his Choice which to exterminate, if one of the three unknown Quantities is not multiplied, or divided by either of the other two, but these are multiplied, or divided by one another, then it will be easiest to find the Value of that unknown Quantity, which is not multiplied, or divided by the others,

Or if one of the unknown Quantities should be to the first Power only in all the three given Equations, and the other two are raised to some higher Power, then it may be easiest to exterminate the unknown Quantity, which is to the first Power only.

And if all the three unknown Quantities are only to the first Power, and none of them are multiplied or divided by one another, then if one of them has no Co-efficient but *Unity*, and the other two have Co-efficients, it may be easiest to exterminate that unknown Quantity, whose Co-efficient is *Unity*.

These Directions may be of Use to the Learner, in assisting his Choice which unknown Quantity to exterminate, and a little Care and Attention will help his Judgment in this Part of the Science; I shall only just mention, that if any particular Difficulties arise from the exterminating one unknown Quantity, it may not be improper to make an Essay how the Work will proceed, from exterminating some other unknown Quantity.

Question 79. *There are three Numbers whose Sum is 18:*

*The first being added to three times the second, from which Sum subtracting twice the third, the Remainder is 9:*

*But if the first is added to four times the third, from which Sum subtracting twice the second, the Remainder is 21. What are the three Numbers?*

Let  $a$  = the first Number,  $e$  = the second Number,  $y$  = the third Number,  $b = 18$ ,  $m = 9$ ,  $p = 21$ .

$$\begin{array}{r|l}
 1 & a + e + y = b \\
 2 & a + 3e - 2y = m \\
 3 & a + 4y - 2e = p \\
 \hline
 1 - y & 4 \quad a + e = b - y \\
 4 - e & 5 \quad a = b - y - e
 \end{array} \left. \vphantom{\begin{array}{r|l} 1 \\ 2 \\ 3 \end{array}} \right\} \text{By the Question.}$$

Having found the Value of  $a$  in the first Equation, in the room of  $a$  in the second and third Equations, put its Value  $b - y - e$ , thus.

$$\begin{array}{r|l}
 2 \cdot 5 & 6 \quad b - y - e + 3e - 2y = m \\
 3 \cdot 5 & 7 \quad b - y - e + 4y - 2e = p \\
 \hline
 6 \text{ contracted} & 8 \quad b - 3y + 2e = m \\
 7 \text{ contracted} & 9 \quad b + 3y - 3e = p
 \end{array} \left. \vphantom{\begin{array}{r|l} 2 \cdot 5 \\ 3 \cdot 5 \end{array}} \right\} \text{Here the Question is reduced to two Equations, and two unknown Quantities, for } a \text{ is exterminated.}$$

Now

Now find the Value of either  $y$ , or  $e$ , in each of these two last Equations, and  $3 y$  being in each Equation, find what that is equal to.

$$\begin{array}{l|l|l} 8 + 3y & 10 & b + 2e = m + 3y \\ 10 - m & 11 & 3y = b + 2e - m \\ 9 + 3e & 12 & b + 3y = p + 3e \\ 12 - b & 13 & 3y = p + 3e - b \\ 11 + 13 & 14 & p + 3e - b = b + 2e - m \end{array}$$

Here we have an Equation with one unknown Quantity only, which is reduced in the common Manner, thus,

$$\begin{array}{l|l|l} 14 - 2e & 15 & p + e - b = b - m \\ 15 + b & 16 & p + e = 2b - m \\ 16 - p & 17 & e = 2b - m - p = 6, \text{ then} \\ 13 \div 3 & 18 & y = \frac{p + 3e - b}{3} = 7, \text{ and} \\ \text{By the fifth Step} & 19 & a = b - y - e = 5. \end{array}$$

Hence the three Numbers sought are 5. 6. and 7.

### P R O O F.

$$\begin{array}{l} a + e + y = 18 \\ a + 3e - 2y = 9 \\ a + 4y - 2e = 21. \end{array}$$

Question 80. *Three Men, A, B, C, discoursing of their Shillings, found, that if twice A's Shillings was added to B's Shillings, and from that Sum subtracting C's Shillings, there remains 15:*

*And if B's Shillings was added to three times C's Shillings, and from that Sum subtracting A's Shillings, there remains 31:*

*But if six times A's Shillings was added to four times C's Shillings, and this Sum added to B's Shillings, the Sum was 97. How many Shillings had each Person?*

Let  $a$  = the Number of Shillings of A,  $e$  = those of B, and  $y$  = those of C,  $b = 15$ ,  $d = 31$ ,  $m = 97$ .

$$\left. \begin{array}{l|l} 1 & 2a + e - y = b \\ 2 & e + 3y - a = d \\ 3 & 6a + 4y + e = m \end{array} \right\} \text{By the Question.}$$

Because

Because  $e$  has no Co-efficient but *Unity*, begin with finding the Value of  $e$ , as being the most simple.

$$\begin{array}{l|l} 1 + y & 4 \\ 4 - 2a & 5 \end{array} \left| \begin{array}{l} 2a + e = b + y \\ e = b + y - 2a \end{array} \right.$$

Now in the second and third Equations, in the room of  $e$ , put its Value, or  $b + y - 2a$ , as in the last Question.

$$\begin{array}{l|l} 2 \cdot 5 & 6 \\ 3 \cdot 5 & 7 \end{array} \left| \begin{array}{l} b + y - 2a + 3y - a = d \\ b + y - 2a + 6a + 4y = m \end{array} \right.$$

Here the Question is reduced to two Equations, and two unknown Quantities,  $e$  being exterminated, and therefore proceeding, as in the last Question,

$$\begin{array}{l|l} 6 \text{ contracted} & 8 \\ 7 \text{ contracted} & 9 \end{array} \left| \begin{array}{l} b + 4y - 3a = d \\ 4a + 5y + b = m \end{array} \right.$$

Now find the Value of either of the unknown Quantities in both these Equations: To find the Value of  $y$ ,

$$\begin{array}{l|l} 8 + 3a & 10 \\ 10 - b & 11 \\ 11 - \overline{4} & 12 \\ 9 - b & 13 \\ 13 - 4a & 14 \\ 14 - \overline{5} & 15 \\ 12 \cdot 15 & 16 \end{array} \left| \begin{array}{l} b + 4y = d + 3a \\ 4y = d + 3a - b \\ y = \frac{d + 3a - b}{4} \\ 4a + 5y = m - b \\ 5y = m - b - 4a \\ y = \frac{m - b - 4a}{5} \\ \frac{d + 3a - b}{4} = \frac{m - b - 4a}{5} \end{array} \right.$$

Here we have an Equation with only the unknown Quantity  $a$ .

$$\begin{array}{l|l} 16 \times 4 & 17 \\ 17 \times 5 & 18 \\ 18 + 16a & 19 \\ 19 + 5b & 20 \\ 20 - 5d & 21 \\ 21 \div 31 & 22 \end{array} \left| \begin{array}{l} d + 3a - b = \frac{4m - 4b - 16a}{5} \\ 5d + 15a - 5b = 4m - 4b - 16a \\ 5d + 31a - 5b = 4m - 4b \\ 5d + 31a = 4m + b \\ 31a = 4m + b - 5d \\ a = \frac{4m + b - 5d}{31} = 8, \text{ then} \end{array} \right.$$



$$\begin{array}{l} \text{By the 12th Step} \\ \text{By the fifth Step} \end{array} \left| \begin{array}{l} 23 \\ 24 \end{array} \right| \begin{array}{l} y = \frac{d + 3a - b}{4} = 10, \text{ and} \\ e = b + y - 2a = 9 \end{array}$$

P R O O F.

$$\begin{array}{r} 2a + e - y = 15 \\ e + 3y - a = 31 \\ 6a + 4y + e = 97 \end{array}$$

I have done these two Questions without putting Letters for the given Numbers, it being more easy and familiar; but now to do the last universally, let us put Letters for the Numbers 2 . 3 . 6 and 4 which are given in the Question, and comparing the former Operation with the following, may render it more easy; but if the Learner finds this too perplexing, he may neglect it, and proceed to the next.

Let  $a . e$  and  $y$  be the three unknown Numbers as before, and  $x = 2, z = 3, s = 6, p = 4$ , then,

$$\left. \begin{array}{l} 1 \left| \begin{array}{l} xa + e - y = b \\ e + zy - a = d \\ sa + py + e = m \end{array} \right. \end{array} \right\} \text{By the Question.}$$

Because  $e$  has no *specious* Co-efficient in either of the given Equations, find the Value of  $e$ .

$$\begin{array}{l} 1 + y \\ 4 - xa \end{array} \left| \begin{array}{l} 4 \\ 5 \end{array} \right| \begin{array}{l} xa + e = b + y \\ e = b + y - xa \end{array}$$

Now in the second and third Equations, in the room of  $e$  put its Value, or  $b + y - xa$ ,

$$\begin{array}{l} 2 \cdot 5 \\ 3 \cdot 5 \end{array} \left| \begin{array}{l} 6 \\ 7 \end{array} \right| \begin{array}{l} b + y - xa + zy - a = d \\ sa + py + b + y - xa = m \end{array}$$

Here the Question is reduced to two Equations, and two unknown Quantities,  $e$  being exterminated; but because of the *specious* Co-efficients we cannot contract them as before: Now find the Value of  $y$ , in both these Equations.

6 + a	8	$b + y - xa + zy = d + a$	
8 + xa	9	$b + y + zy = d + a + xa$	
9 - b	10	$zy + y = d + a + xa - b$	
10 ÷ z + 1	11	$y = \frac{d + a + xa - b}{z + 1}$	the Co-efficients of $y$ being ( $z + 1$ )
7 + xa	12	$sa + py + b + y = m + xa$	
12 - b	13	$sa + py + y = m + xa - b$	
13 - sa	14	$py + y = m + xa - b - sa$	
14 ÷ p + 1	15	$y = \frac{m + xa - b - sa}{p + 1}$	the Co-efficients of $y$ being ( $p + 1$ )
11 . 15	16	$\frac{d + a + xa - b}{z + 1} = \frac{m + xa - b - sa}{p + 1}$	an Equation
		(with only the unknown Quantity $a$ .	
16 × z + 1	17	$d + a + xa - b = \frac{zm + zxa - zb - zsa + m + xa - b - sa}{p + 1}$	
17 × p + 1	18	$pd + pa + pxa - pb + d + a + xa - b = zm + zxa - zb - zsa + m + xa - b - sa$	

The Learner may think these Multiplications discouraging, though perhaps they are not so perplexing as he may imagine, for at the seventeenth Step where  $m + xa - b - sa$  is  $xz + 1$ , put down the Product of it by  $z$  first, which is  $zm + zxa - zb - zsa$ , after which he need only write  $m + xa - b - sa$ , the next Part of the Multiplier being *Unity*; or, if it had been another Letter, it had been no more than repeating the *Multiplicand*, with the *multiplying Letter* joined to each of its Quantities, placing them one after another, taking due Care of the Signs by the Rules for Multiplication.

In the same Manner he will find the eighteenth Step multiplied, and a little Attention will familiarize the Operation; but if there is any Difficulty in multiplying these compound Quantities, the Learner may set them down one under the other, and multiply them in the usual Manner.

18 - xa	19	$pd + pa + pxa - pb + d + a - b = zm + zxa$ $- zb - zsa + m - b - sa$
19 + b	20	$pd + pa + pxa - pb + d + a = zm + zxa - zb$ $- zsa + m - sa$

Now transpose all the unknown Quantities to one Side of the Equation, and all the known ones to the other Side of the Equation.

20 — $z x a$	21	$p d + p a + p x a - p b + d + a - z x a$ $= z m - z b - z s a + m - s a$
21 + $z s a$	22	$p d + p a + p x a - p b + d + a + z s a$ $- z x a = z m - z b + m - s a$
22 + $s a$	23	$p d + p a + p x a - p b + d + a + z s a$ $- z x a + s a = z m - z b + m$
23 — $p d$	24	$p a + p x a - p b + d + a + z s a - z x a$ $+ s a = z m - z b + m - p d$
24 + $p b$	25	$p a + p x a + d + a + z s a - z x a + s a$ $= z m - z b + m - p d + p b$
25 — $d$	26	$p a + p x a + a + z s a - z x a + s a$ $= z m - z b + m - p d + p b - d$
26 ÷	27	$a = \frac{z m - z b + m - p d + p b - d}{p + p x + 1 + z s - z x + s} = 8,$ the Divisor is the Co-efficients of $a$ , connected by their Signs.
Then by 11th Step	28	$y = \frac{d + a + x a - b}{z + 1} = 10$
And by 5th Step	29	$e = b + y - x a = 9$

Question 81. *There are three Travellers, A, B, C, who have travelled in all 62 Miles :*

*But if the Miles A travelled is multiplied by 2, and added to the Miles B travelled multiplied by 3, this Sum is equal to the Miles C travelled multiplied by 17 :*

*And if 4 times the Miles C travelled, is added to the Miles B travelled multiplied by 2, this Sum is equal to the Miles travelled by A. To find the Miles each travelled ?*

Let  $a$  = the Miles travelled by A,  $e$  = the Miles travelled by B,  $y$  = the Miles travelled by C;  $p = 62$ ,  $b = 2$ ,  $d = 3$ ,  $m = 17$ ,  $x = 4$ , and 2 being in the Question before, put no new Letter for it.

$$\left. \begin{array}{l} 1 \quad a + e + y = p \\ 2 \quad b a + d e = m y \\ 3 \quad x y + b e = a \end{array} \right\} \text{By the Question.}$$

Because  $a$  seems to be in as simple Terms as any in the three given Equations, and having its Value already by the third Equation, therefore for  $a$  in the first and second Equation write its Value  $x y + b e$  at the third Equation, which exterminates  $a$ .

$$\begin{array}{l|l} 1.3 & 4 \\ 2.3 & 5 \end{array} \left| \begin{array}{l} xy + be + e + y = p \\ bxy + bbe + de = my \end{array} \right.$$

Here the Question is reduced to two Equations, and two unknown Quantities, then proceed, as before, to find the Value of either  $y$  or  $e$ , in each of these Equations, as suppose  $y$ .

$$\begin{array}{l|l} 4 - be & 6 \\ 6 - e & 7 \end{array} \left| \begin{array}{l} xy + e + y = p - be \\ xy + y = p - be - e \end{array} \right.$$

$$7 \div x + 1 \quad 8 \quad y = \frac{p - be - e}{x + 1} \text{ the Co-efficients of } y \text{ being } x + 1.$$

$$5 - bxy \quad 9 \quad my - bxy = bbe + de$$

$$9 \div m - bx \quad 10 \quad y = \frac{bbe + de}{m - bx} \text{ the Co-efficients of } y \text{ being } m - bx.$$

$$8 \cdot 10 \quad 11 \quad \frac{bbe + de}{m - bx} = \frac{p - be - e}{x + 1} \text{ an Equation with only one unknown Quantity.}$$

$$11 \times m - xb \quad 12 \quad bbe + de = \frac{mp - mbe - me - pbx + bbe + bxe}{x + 1}$$

$$12 \times x + 1 \quad 13 \quad xbbe + xde + bbe + de = mp - mbe - me - pbx + bbe + bxe$$

$$13 - xbbe \quad 14 \quad xde + bbe + de = mp - mbe - me - pbx + bxe$$

Now bring the Terms that have the unknown Quantity, to one Side of the Equation.

$$14 + mbe \quad 15 \quad xde + bbe + de + mbe = mp - me - pbx + bxe$$

$$15 + me \quad 16 \quad xde + bbe + de + mbe + me = mp - pbx + bxe$$

$$16 - bxe \quad 17 \quad xde + bbe + de + mbe + me - bxe = mp - pbx$$

$$17 \div \quad 18 \quad e = \frac{mp - pbx}{xd + bb + d + mb + m - bx} = 9, \text{ the Divisor is the Co-efficients of } e, \text{ connected by their Signs.}$$

$$\begin{array}{l} \text{By Step } 8, \text{ or } 10. \end{array} \left. \begin{array}{l} 19 \\ 20 \end{array} \right\} \begin{array}{l} y = 7 \\ a = 46 \end{array}$$

Question 82. *Three Men, A, B, C, discoursing of their Shillings, found, that A's Shillings added to C's Shillings, the Sum was double B's Shillings:*

*And A's Shillings added to three times B's Shillings, from which Sum subtracting C's Shillings, there remained 13 Shillings:*

*But if A's Shillings was added to the Product of B's and C's Shillings, the Sum was 34. How many Shillings had each Person?*

Let

Let  $a =$  A's Shillings,  $e =$  B's Shillings,  $y =$  C's Shillings,  
 $b = 13$ ,  $d = 34$ .

	1	$a + y = 2e$	} By the Question.
	2	$a + 3e - y = b$	
	3	$a + ey = d$	
1 - y	4	$a = 2e - y$	
2 . 4	5	$2e - y + 3e - y = b$	} Here the Question is reduced to two Equations and two unknown Quantities, $a$ being exterminated.
3 . 4	6	$2e - y + ey = d$	
5 contracted	7	$5e - 2y = b$	

Now find the Value of  $e$ , or  $y$ , in the sixth and seventh Equations, suppose  $e$ .

$6 + y$	8	$2e + ey = d + y$	
$8 \div 2 + y$	9	$e = \frac{d + y}{2 + y}$	
$7 + 2y$	10	$5e = b + 2y$	
$10 \div 5$	11	$e = \frac{b + 2y}{5}$	
$9 . 11$	12	$\frac{b + 2y}{5} = \frac{d + y}{2 + y}$ , an Equation with only one unknown Quantity.	
$12 \times 2 + y$	13	$\frac{2b + 4y + by + 2yy}{5} = d + y$	
$13 \times 5$	14	$2b + 4y + by + 2yy = 5d + 5y$	

Now bring all the Quantities that have  $y$ , to one Side of the Equation.

$14 - 5y$	15	$2b - y + by + 2yy = 5d$
$15 - 2b$	16	$2yy + by - y = 5d - 2b$

Here the Equation appears *quadratic*, the unknown Quantity being to the second and first Power only, but is not ambiguous,  $5d$  being greater than  $2b$ ; then by Art. 58, divide by the Co-efficient of  $yy$ .

$16 \div 2$	17	$yy + \frac{by - y}{2} = \frac{5d - 2b}{2}$
		G g 2

The

The Work being now prepared for *completing the Square*, because the Co-efficient of  $y$  is  $\frac{b-1}{2}$ , to avoid the Trouble of dividing this Fraction by 2, and squaring the Quotient, substitute by Art. 57.  $x = \frac{b-1}{2} = 6$ .

Then	18	$yy + xy = \frac{5d - 2b}{2}$
18 c □	19	$yy + xy + \frac{xx}{4} = \frac{5d - 2b}{2} + \frac{xx}{4}$
19 w 2	20	$y + \frac{x}{2} = \sqrt{\frac{5d - 2b}{2} + \frac{xx}{4}}$
20 - $\frac{x}{2}$	21	$y = \sqrt{\frac{5d - 2b}{2} + \frac{xx}{4}} - \frac{x}{2} = 6,$ (C's Shillings.)
By the 9th, or } 11th Steps }	22	$e = 5,$ B's Shillings.
By the 4th Step	23	$a = 4,$ A's Shillings.

Question 83. *Three young Gentlemen, A, B, C, having been at the Gaming-Tables, from comparing their Losses, found, that if from twice the Pounds A lost, was subtracted the Pounds B lost, there remained the Pounds C lost :*

*And that the Pounds A lost, added to the Pounds B lost, and this Sum added to twice the Pounds C lost, the Sum was 19 Pounds :*

*But if to the Product of A's and C's Loss, there is added B's Loss, the Sum is 26 Pounds. How much did each Person lose ?*

Let  $a =$  A's Loss,  $e =$  B's Loss,  $y =$  C's Loss,  $d = 19$ ,  $b = 26$ .

$$\left. \begin{array}{l|l} 1 & 2a - e = y \\ 2 & a + e + 2y = d \\ 3 & ay + e = b \end{array} \right\} \text{By the Question.}$$

Because  $e$  seems to be in the most simple Terms, therefore find its Value.

$$\begin{array}{l|l} 1 + e & 4 \\ 4 - y & 5 \end{array} \left| \begin{array}{l} 2a = y + e \\ e = 2a - y \end{array} \right.$$

$$\begin{array}{l}
 2 \cdot 5 \left| \begin{array}{l} 6 \\ 3 \cdot 5 \end{array} \right| \begin{array}{l} a + 2a - y + 2y = d \\ ay + 2a - y = b \end{array} \\
 6 \text{ contracted } \left| \begin{array}{l} 8 \\ 3 \end{array} \right| \begin{array}{l} a + 2a - y + 2y = d \\ ay + 2a - y = b \end{array} \\
 \end{array}
 \left. \vphantom{\begin{array}{l} 2 \cdot 5 \\ 3 \cdot 5 \\ 6 \text{ contracted} \end{array}} \right\} \begin{array}{l} \text{The Question is here} \\ \text{reduced to two Equations and two unknown} \\ \text{Quantities, for } c \text{ is ex-} \\ \text{terminated.} \end{array}$$

Find the Value of  $a$ , or  $y$ , in the seventh and eighth Equations.

$$\begin{array}{l}
 7 - 2a \left| \begin{array}{l} 9 \\ 10 \end{array} \right| \begin{array}{l} ay - y = b - 2a \\ y = \frac{b - 2a}{a - 1} \end{array} \\
 9 \div a - 1 \left| \begin{array}{l} 11 \\ 12 \end{array} \right| \begin{array}{l} y = d - 3a \\ \frac{b - 2a}{a - 1} = d - 3a \end{array} \\
 8 - 3a \left| \begin{array}{l} 13 \\ 14 \end{array} \right| \begin{array}{l} b - 2a = da - 3a - d + 3a \\ 12 \times a - 1 \end{array} \\
 10 \cdot 11 \left| \begin{array}{l} 15 \\ 16 \end{array} \right| \begin{array}{l} 3aa + b - 2a = da - d + 3a \\ 3aa + b - 5a = da - d \end{array} \\
 12 \times a - 1 \left| \begin{array}{l} 17 \\ 18 \end{array} \right| \begin{array}{l} 3aa + b - 5a - da = -d \\ 3aa - 5a - da = -d - b \end{array}
 \end{array}$$

Now bring all the Quantities that have  $a$  on one Side of the Equation, observing to have the highest Power of  $a$  affirmative.

$$\begin{array}{l}
 13 + 3aa \left| \begin{array}{l} 14 \\ 15 \end{array} \right| \begin{array}{l} 3aa + b - 2a = da - d + 3a \\ 3aa + b - 5a = da - d \end{array} \\
 14 - 3a \left| \begin{array}{l} 16 \\ 17 \end{array} \right| \begin{array}{l} 3aa + b - 5a - da = -d \\ 3aa - 5a - da = -d - b \end{array} \\
 15 - da \left| \begin{array}{l} 18 \\ 19 \end{array} \right| \begin{array}{l} 3aa - 5a - da = -d - b \\ 3aa - 5a - da = -d - b \end{array} \\
 16 - b \left| \begin{array}{l} 19 \\ 20 \end{array} \right| \begin{array}{l} 3aa - 5a - da = -d - b \\ 3aa - 5a - da = -d - b \end{array}
 \end{array}$$

Here the Equation appears both *quadratic* and *ambiguous*, for the unknown Quantity is to the second and first Power only, and it is *ambiguous*, because  $-d - b$  the Side of the Equation which is known, is *negative*; dividing by the Co-efficient of  $aa$ , as in the last Question,

$$17 \div 3 \left| \begin{array}{l} 18 \\ 19 \end{array} \right| \begin{array}{l} aa - \frac{5a - da}{3} = \frac{-d - b}{3} \\ aa - xa = \frac{-d - b}{3} \end{array}$$

The Work being now prepared for completing the Square, substitute  $-x = \frac{-5 - d}{3} = -8$  the Co-efficients of  $a$ , as in the last Example.

$$\begin{array}{l}
 \text{Then } \left| \begin{array}{l} 19 \\ 20 \end{array} \right| \begin{array}{l} aa - xa = \frac{-d - b}{3} \\ aa - xa + \frac{xx}{4} = \frac{-d - b}{3} + \frac{xx}{4} \end{array} \\
 18 \text{ c } \square \left| \begin{array}{l} 20 \\ 21 \end{array} \right| \begin{array}{l} aa - xa + \frac{xx}{4} = \frac{-d - b}{3} + \frac{xx}{4} \\ aa - xa + \frac{xx}{4} = \frac{-d - b}{3} + \frac{xx}{4} \end{array}
 \end{array}$$

$$\begin{array}{l|l}
 19 \text{ or } 2 & 21 \\
 20 + \frac{x}{2} & 22
 \end{array}
 \left| \begin{array}{l}
 a - \frac{x}{2} = \sqrt{\frac{-d-b}{3} + \frac{xx}{4}} \\
 a = \frac{x}{2} \pm \sqrt{\frac{-d-b}{3} + \frac{xx}{4}} = 4,
 \end{array} \right.
 (\pm 1 = 3, \text{ or } 5.$$

For the Practice of the Learner, let us suppose  $a = 3$

Then by the tenth, or eleventh Steps -  $y = 10$

And by the fifth Step -  $e = 6 - 10$   
 $= -4$ , which is an Impossibility, that  $e$  an affirmative Quantity, can be equal to a negative 4.

Now let us suppose -  $a = 5$

Then by the tenth, or eleventh Steps -  $y = 4$

And by the fifth Step -  $e = 6$

Then  $2a - e = y$

$$a + e + 2y = 19$$

$$ay + e = 26$$

And these three Numbers answering the Conditions of the Question, are the true Numbers sought; from hence the young Analyst may observe, that in *quadratic ambiguous* Equations, if one of the Roots of the unknown Quantity does not answer the Conditions of the Question, he should find the other Root, and try that, before he concludes his Work erroneous.

I shall now show the Learner the excellent Method of resolving all Equations, be their Powers never so high, by the universal Method of *Converging Series*.

## 67. The Resolution of Affected Equations, by the universal Method of Converging Series.

### C A S E I.

Ex. 1. SUPPOSE there was given  $aaa + a = 9282$ , to find  $a$

Then suppose, or imagine  $a$  to be -  $20$

Consequently the Cube of  $a$ , or  $aaa$ , is -  $8000$

These being added together, because it is  $aaa + a$  }  $8020$   
 in the given Equation, the Sum is -



Hence  $a$  must be more than 20, for if that had been the true Root, the Cube of 20 added to its first Power, or 20, must have been equal to 9282 the given Number, for these are the same Powers of  $a$  as in the given Equation; but that Sum being only 8020, which being less than 9282, the Value of  $a$  must be more than 20. To find how much that is?

Let  $r = 20$ , and for what 20 wants of the true Number or Root, put  $e$ :

Then will  $r + e = a$ , or the true Root of the Equation, hence by determining what  $e$  is, we find the Number that is to be added to  $r$  or 20, which Sum will be the Root of the given affected Equation, to do which, put down,

$$1 \mid r + e = a$$

Now raise this Equation to the third Power, because we have  $aaa$  in the given Equation.

$$1 \oplus 3 \mid 2 \mid rrr + 3rre + 3ree + eee = aaa$$

Add the first and second Equations together, because in the given Equation it is  $aaa + a$ .

$$1 + 2 \mid 3 \mid rrr + 3rre + 3ree + eee + r + e = aaa + a$$

But from the given Equation

$$aaa + a = 9282$$

$$3 \cdot 4 \mid 5 \mid rrr + 3rre + 3ree + eee + r + e = 9282$$

each Equation being equal to  $aaa + a$ .  
Putting this Equation into Numbers,  
and rejecting the Powers of  $e$  above  $ee$ .

$$5 \text{ in Numbers} \mid 6 \mid 8000 + 1200e + 60ee + 20 + e = 9282$$

$$\text{That is} \mid 7 \mid 8020 + 1201e + 60ee = 9282$$

$$7 - 8020 \mid 8 \mid 1201e + 60ee = 1262$$

Divide by the Co-efficient of  $ee$ , and we have,

$$7 \div 60 \mid 9 \mid 20.016e + ee = 21.0333 \text{ ad infinitum.}$$

Dividing by  $20.016 + e$ , that is, by the Co-efficient of  $e$  plus  $e$ , and we have,

$$9 \div 20.016 + e \mid 10 \mid e = \frac{21.0333}{20.016 + e}$$

In Numbers thus:

$$20.016 \mid 21.0333 \ (1 = e)$$

1

$$\begin{array}{r} 21016 \\ 21.016 \end{array}$$

17 the Remainder  
(being very small reject it.  
By

For the Reason of adding the Quotient Figure, or 1, to the Divisor, see Article 68, in the next Page.

By this it appears that  $e = 1$ , that is, 1 is to be added to the first supposed Number 20, which Sum is to be the Value of  $a$ , or the Root of the given affected Equation.

$$\begin{array}{r} \text{We assumed } r = 20 \\ \text{And found } e = 1 \\ \hline r + e = 21 = a \end{array}$$

To try whether 21 is the true Root, raise it to the several Powers of  $a$  in the given Equation.

$$\begin{array}{r} a = 21 \\ a = 21 \\ \hline 21 \\ 42 \\ \hline aa = 441 \\ a = 21 \\ \hline 441 \\ 882 \\ \hline aaa = 9261 \\ a = 21 \end{array}$$

Then  $aaa + a = 9282$ , which being the same with the Number in the given Equation, it appears that  $a = 21$ .

68. By reviewing the Operation, the Learner may observe, *First*, That we supposed a Number for the true Root, which upon Trial was found less than the true Root.

*Secondly*, For that Deficiency or Want, we put  $e$ , or any other Letter.

*Thirdly*, By connecting  $r =$  the Number first supposed to be the Root, with  $e$  by the Sign  $+$ , we have  $r + e$  for the true Root,  $r$  being a known Quantity, and  $e$  the unknown Quantity.

*Fourthly*, We raise  $r + e$  to the several Powers of the unknown Quantity, that are in the given affected Equation.

*Fifthly*, Then we add these several Equations together, rejecting all the Powers of  $e$ , or of the unknown Quantity above the Square, for in the given Equation all the Powers of the unknown Quantity have the Sign  $+$ , but when any of these have the Sign  $-$ , then their respective Equations must be subtracted, as at Step 5, Example 3, Page 238.

*Sixthly*, By these Means we have an Equation in the Terms of  $r$  and  $e$ , equal to the given Equation.

*Seventhly*,

*Seventhly*, This Equation is put into Numbers,  $r$  being a known Quantity, and the less absolute Number is transposed to the Side of the Equation of the greater absolute Number, and subtracted from it.

*Eighthly*, After this, the Equation is divided by the Co-efficient of the Square of  $e$ , or the unknown Quantity.

*Ninthly*, This last Equation is divided by the Co-efficient of  $e$  plus  $e$ , which leaves  $e$  on one Side of the Equation by itself.

*Tenthly*, In the Arithmetical Work, because the last Divisor consists of a Number plus  $e$ , therefore, as the Quotient Figure is found, it is added to the Divisor to make it compleat; and if the numerical Operation had been continued to more Places of Figures in the Quotient, then the Quotient Figure must be twice added, once when it is found, and once at the next Step in the Division, as in the next Page.

*Eleventhly*, The Quotient thus found being the Value of  $e$ , or the unknown Quantity, it is added to the Number first supposed to be the Root of the Equation, which is represented by  $r$ , and this Sum is supposed to be the Root required.

This Operation to find  $e$  is the same as the common Method of finding the unknown Quantity, till we come to the tenth Step, where the unknown Quantity making Part of the Divisor, it is carried to the other Side of the Equation, and the Divisor being a known Number +  $e$ , the Quotient as it is found is added to the Divisor, to make it compleat, as before-mentioned.

But if this Number should not be the true Root, the Operation must be repeated, making the Number thus found =  $r$ , and at the second Operation, the Work in any common Case will be sufficiently exact: And from the Repetition of the Operation, whereby we approach nearer and nearer to the true Root, this Method is called the Method of *Converging Series*, or of *Approximation*.

*Example 2.* Suppose  $aaa + aa + a = 42997850$ , to find  $a$ .

Suppose $a$ to be	-	-	-	300
Then the Cube of $a$ , or $aaa$ is	-	-	-	27000000
And the Square of $a$ , or $aa$ is	-	-	-	90000
				27090300
These being added, because it is $aaa + aa + a$				27090300
in the given Equation, the Sum is	-	-	-	

But 42997850, the given Number, is greater than 27090300, therefore the Root must be more than 300.

H h

Now

Now let  $r = 300$ , and  $e =$  what 300 wants of the true Root.

Then	1	$r + e = a$	} According to Particular 4, Art. 68.
$1 \ominus 3$	2	$rrr + 3rre + 3ree$	
		$+ eee = aaa$	
$1 \ominus 2$	3	$rr + 2re + ee = aa$	} According to Particulars 5 and 6, Art. 68.
$1 + 2 + 3$	4	$r + e + rrr + 3rre + 3ree + rr + 2re + ee$	
		$= aaa + aa + a$	
But	5	$aaa + aa + a = 42997850$ , from the given Equation.	
$4 \cdot 5$	6	$r + e + rrr + 3rre + 3ree + rr + 2re + ee = 42997850$	
6 in Numbers	7	$300 + e + 27000000 + 270000e + 900ee + 90000 + 600e + ee = 42997850$	
That is	8	$27090300 + 270601e + 901ee = 42997850$	
$8 - 27090300$	9	$270601e + 901ee = 15907550$	
$9 \div 901$	10	$300.334e + ee = 17655.43$ from Particular 8, Art. 68.	
$13 \div 300.334 + e$	11	$e = \frac{17655.43}{300.334 + e}$ , from Particular 9, (Art. 68.)	

300.334)	17655.43	(50.34 = $e$ , the first Figure being in the Place of Tens, place the 5 under the Place of Tens in the Divisor; and this the Reader is to observe, to place the Quotient Figures he adds to the Divisor, under those of the same Denomination.
5		
Divisor 350.334	1751670	
50.3		
Divisor 400.634	1387300	
.34	1201902	
	1853980	
Divisor 400.974	1603896	
	250084	

$$r = 300$$

$$e = 50.34$$

$r + e = 350.34 = a$ , hence I suppose the Root of the given Equation is 350.34 but to try it, raise 350.34 to the several Powers of  $a$  in the given Equation.

$$\begin{array}{r}
 a = 350.34 \\
 a = 350.34 \\
 \hline
 140136 \\
 105102 \\
 1751700 \\
 105102 \\
 \hline
 aa = 122738.1156 \\
 a = 350.34 \\
 \hline
 4909524624 \\
 3682143468 \\
 61369057800 \\
 3682143468 \\
 \hline
 aaa = 43000071.419304 \\
 aa = 122738.1156 \\
 a = 350.34 \\
 \hline
 \end{array}$$

Then  $aaa + aa + a = 43123159.874904$  which being greater than 42997850, the Root cannot be so much as 350.34. and this leads us to explain the Method of finding the true Root, when the Number assumed is greater than the Root required, or,

## CASE 2.

Let us take the last *Example*, viz.  $aaa + aa + a = 42997850$ .

And suppose the Root to be 350.34 which we know is too much by the last Operation :

Now put  $e$  = the Number to be subtracted from 350.34 supposing  $r = 350.34$  and to  $r$  connecting  $e$  by the Sign —, we have,

$$\begin{array}{lcl}
 1 \ominus 3 & \left| \begin{array}{l} 1 \\ 2 \end{array} \right. & \left. \begin{array}{l} r - e = a, \text{ or the true Root} \\ rrr - 3rre + 3ree - eee \\ = aaa \end{array} \right\} \begin{array}{l} \text{By Parti-} \\ \text{cular 4.} \\ \text{Art. 68.} \end{array} \\
 1 \ominus 2 & \left| \begin{array}{l} 3 \end{array} \right. & \left. \begin{array}{l} rr - 2re + ee = aa \end{array} \right\}
 \end{array}$$

Now collect these three Equations by Art. 68, Particulars 5 and 6, and rejecting  $eee$ , we have,

$$\begin{array}{lcl}
 1 + 2 + 3 & \left| \begin{array}{l} 4 \end{array} \right. & \left. \begin{array}{l} r - e + rrr - 3rre + 3ree + rr - 2re \\ + ee = aaa + aa + a \end{array} \right. \\
 \text{But} & \left| \begin{array}{l} 5 \end{array} \right. & \left. \begin{array}{l} aaa + aa + a = 42997850, \text{ from the} \\ \text{given Equation.} \end{array} \right. \\
 4 \cdot 5 & \left| \begin{array}{l} 6 \end{array} \right. & \left. \begin{array}{l} r - e + rrr - 3rre + 3ree + rr - 2re \\ + ee = 42997850 \end{array} \right.
 \end{array}$$

H h 2

6 in

6 in Num.	7	$350.34 - e + 43000071.42 - 368214.3468e$ $+ 1051.02ee + 122738.1156 - 700.68e$ $+ ee = 42997850$
That is	8	$43123159.8756 - 368916.0268e + 1052.02ee$ $= 42997850$ Now transpose $42997850$ , it being less than $43123159.8765$
8 —	9	$125309.8756 - 368916.0268e + 1052.02ee$ $= 0$ , for one Side of the Equation being subtracted from the other Side, the Re- mainder must be nothing, as both Sides of the Equation are equal. Now transpose the several Quantities which contain $e$ to the other Side of the Equation.
9 +	10	$125309.8756 + 1052.02ee = 368916.0268e$
10 —	11	$368916.0268e - 1052.02ee = 125309.8756$ Here dividing by the Co-efficient of $ee$ as before,
11 ÷	12	$350.673e - ee = 119.1135$ But now divide by the Co-efficient of $e$ <i>minus e.</i>
12 ÷	13	$e = \frac{119.1135}{350.673 - e}$

In Numbers :

$$\begin{array}{r}
 350.673) 119.1135 (.34 \\
 \underline{-.3} \\
 \text{Divisor } 350.373 \quad 1051119 \\
 \underline{-.34} \\
 1400160 \\
 \text{Divisor } 350.033 \quad 1400132 \\
 \underline{\hspace{1.5cm}} \\
 28
 \end{array}$$

Having thus determined  $e$  to be .34 it must now be subtracted from  $r$ , because it was assumed  $r - e =$  the true Root.

$$\begin{array}{r}
 \text{But } r \text{ was supposed } = 350.34 \\
 - e, \text{ which we have found } = .34 \\
 \hline
 \end{array}$$

Hence  $r - e = 350. = a$ , the Root of the given affected

adfected Equation, which is proved by raising 350. to the feveral Powers of  $a$  in the given Equation,

$$\begin{array}{r} \text{Thus, } a a a = 42875000 \\ a a = 122500 \\ a = 350 \end{array}$$

Consequently  $a a a + a a + a = 42997850$  which being the same Number as in the given Equation, it shows that  $a$  is exactly equal to 350.

In the above Operation, at the thirteenth Step, the Learner may observe, that the Divisor is  $350.673-e$ , therefore here, as the Quotient Figure is found, we *subtract* it from the Part of the Divisor 350.673 to have the Divisor compleat, which is likewise done twice, once before the Division is made at that Figure, and once afterwards: But in the first *Case*, when  $r$  is assumed too little, then the Quotient Figure is added, as at Particular 10, Art. 68. the Sign then being contrary to what it is now.

The Learner may further observe, that by this second Operation, we have found the *true Root*, whereas by the first Operation it was .34 too much, and therefore if the true Root does not come out at the first Operation, make a second Operation, supposing the Number found at the first Operation to be  $r$ , and call it  $r + e$ , or  $r - e$ , for the true Root as the Occasion requires, that is, as the Number at the first Operation is either greater or lesser than the true Root; which second Operation will give the true Root very near, and near enough for any common *Case*, though if the Arithmetical Divisions were continued, as they will not terminate, do not give the true Root exactly, as in the Division of those Decimal Fractions which never terminate; in such Divisions we leave off when the Quotient is to a sufficient Degree of Exactness, so the same is done here when we are near enough the Truth; and in common Cases, two or three Places of Decimal Fractions are sufficient, and according as they happen the true Root is sometimes found; and in general, continue the Division, at the *second Operation*, to as many Places of Decimal Fractions as are in the Number found in the *first Operation*: And after the Number found at the second Operation is added to, or subtracted from the Number found at the first Operation, if there is a very small Fraction you may reject it; but if the Fraction should be very near an *Unit*, then take 1 for it, which add to the Integers, and try whether the whole Number thus found is not the true Root. In Arithmetical Questions, whose Answers are  
often





Putting this Equation in Numbers, and rejecting all the Powers of  $e$  above  $ee$ .

7 in Numbers	8	$400 - e + 64000000 - 480000e +$ $1200ee - 160000 + 800e - ee =$ $46526760$
8 contracted	9	$63840400 + 1199ee - 479201e =$ $46526760$ Transpose $46526760$ it being less than $63840400$
9 —	10	$17313640 + 1199ee - 479201e = 0$ Now transpose the Quantities that have $e$ , to the other Side of the Equation.
10 +	11	$479201e = 17313640 + 1199ee$
11 —	12	$479201e - 1199ee = 17313640$ Dividing by the Co-efficient of $ee$ ,
12 ÷	13	$399.66e - ee = 14440.06$ Now dividing by the Co-efficient of $e$ <i>minus <math>e</math>.</i>
13 ÷	14	$e = \frac{14440.06}{399.66 - e}$

In Numbers thus:

$$\begin{array}{r} 399.66) 14440.06 \quad (40.16 = e. \\ \underline{40.} \end{array}$$

Divisor	359.66	143864	
	40.1		
			53660
Divisor	319.56		31956
	.16		217040
Divisor	319.40		191640
			25490

Now  $r = 400$

$$\underline{- e = 40.16}$$

$r - e = 359.84 = a$ , and to try if this is the true Root, raise it to the several Powers of  $a$ , in the given Equation.

$a =$

$$\begin{array}{r}
 a = 359.84 \\
 a = 359.84 \\
 \hline
 143936 \\
 287872 \\
 323856 \\
 179920 \\
 107952 \\
 \hline
 aa = 129484.8256 \\
 a = 359.84 \\
 \hline
 5179393024 \\
 10358786048 \\
 11653634304 \\
 6474241280 \\
 3884544768 \\
 \hline
 aaa = 46593819.643904 \\
 - aa = 129484.8256 \\
 \hline
 \text{Remains } 46464334.818304 \\
 + a = 359.84 \\
 \hline
 \end{array}$$

Sum, or  $aaa - aa + a = 46464694.658304$ , which being less than 46526760 the Number in the given Equation, the Root or  $a$  must be more than 359.84.

Therefore, for a second Operation, suppose  $r = 359.84$  and  $e =$  what it wants of the true Root, then it being  $r + e = a$ , it is now the *first Case*, Page 230.

Therefore	1	$r + e = a$
1 $\ominus$ 3	2	$rrr + 3rre + 3ree + eee = aaa$
1 $\ominus$ 2	3	$rr + 2re + ee = aa$
		Add the first and second Equation together, because in the given Equation it is $a + aaa$ .
1 + 2	4	$rrr + 3rre + 3ree + eee + r + e = aaa + a$ From this Equation subtract the third Equation, because it is $-aa$ in the given Equation.
4 - 3	5	$rrr + 3rre + 3ree + eee + r + e - rr - 2re - ee = aaa - aa + a$
But	6	$aaa - aa + a = 46526760$ by the given Equation.
5 . 6	7	$rrr + 3rre + 3ree + eee + r + e - rr - 2re - ee = 46526760$

		Put this Equation in Numbers, and reject the Powers of $e$ , above $ee$ .
7 in Numbers	8	$46593819.644 + 388454.4768e + 1079.52ee + 359.84 + e - 129484.8256 - 719.68e - ee = 46526760$
8 contracted	9	$46464694.6584 + 387735.7968e + 1078.52ee = 46526760$
9 —	10	$387735.7968e + 1078.52ee = 62065.3416$
		Dividing by the Co-efficient of $ee$ .
10 ÷	11	$359.5e + ee = 57.547$
		Now dividing by the Co-efficient of $e$ plus $e$ .
11 ÷	12	$e = \frac{57.547}{359.5 + e}$

In Numbers thus :

$$\begin{array}{r} 359.5) 57.547 \quad (.16 = e \\ + .1 \end{array}$$

$$\begin{array}{r} \text{Divisor } 359.6 \quad 3596 \\ .16 \end{array}$$

$$\begin{array}{r} \text{Divisor } 359.76 \quad 215870 \\ 215856 \end{array}$$

14

The Reader will observe that in this Division, I have taken at once two Figures from the Dividend, *viz.* 70, because in adding the .16 to the Divisor, the Number of Places there is increased by one, therefore I take one Figure more from the Dividend than is usual ; which is recommended to the Reader's Attention, as he may again meet with the same Case.

$$\begin{array}{r} \text{Now } r = 359.84 \\ + e = .16 \end{array}$$

$r + e = 360.00 = a$ , which will be found to be the true Root, by involving it to the several Powers of  $a$  in the given Equation, and adding or subtracting them according as those Powers of  $a$  are there connected by the Signs  $+$  or  $-$ .

It may be proper to inform the Learner, that the nearer the Number is taken to the true Root, the nearer the Operation will come to the Truth, and therefore after he has tried the *first Supposition*, if he thinks he can make a second Supposition nearer the Truth, it will be right to do it, which perhaps  
I i  
may

may bring out the Root so near at the first Operation, that it may save him the Trouble of making a second Operation. Thus,

If  $a a a + a a + a = 4942070$ , to find  $a$ .

Suppose  $a$  to be 100.

Then the Cube of $a$ , or $a a a$ is	-	-	-	1000000
And the Square of $a$ , or $a a$ is	-	-	-	10000
And $a$ is	-	-	-	100
				<hr/> 1010100

The Number in the given Equation is - 4942070

If we suppose  $a = 100$ , then the Sum of its } 1010100  
several Powers are - - -

Difference wanting - - - 3931970

Now let us make a second Supposition thus,

If  $a = 200$ ,

Then the Cube of $a$ , or $a a a$ is	-	-	8000000
And the Square of $a$ , or $a a$ is	-	-	40000
And $a$ is	-	-	200
			<hr/> 8040200

Sum of the several Powers of  $a$ , if  $a$  is 200 - 8040200

The Number in the given Equation - - 4942070

Difference over 3098130

For a *third* Supposition, suppose it 160 and try with that, and if it be *less than just*, it must be  $r = 160$  and  $r + e = a$ ; if 160 be too much or *more than just*, then it must be  $r - e = a$ .

When there are two Suppositions made, one being *more than just*, and the other *less than just*, it may be convenient to make a third Supposition between the two, and proceed by *Case 1* or *2*, according as the supposed Number is *more* or *less than just*.

Having explained the Method of resolving affected Equations, we proceed to such Questions as produce these Equations.

*The Manner of solving Questions, when the unknown Quantity has several Powers in one Equation, and only the first Power in the other Equation.*

69. **W**HEN the unknown Quantities are to the first and second Power in one Equation, and but to the first Power in the other Equation, find the Value of that unknown Quantity, in the Equation where its Terms are the more simple; raise this Equation, or Value of the unknown Quantity, to the several Powers of the unknown Quantity in the other Equation; then in that Equation for the several Powers of the unknown Quantity, write, or put these Values, which *exterminates* that unknown Quantity, leaving an Equation with only one unknown Quantity, which may be resolved by some of the Methods already explained.

Question 84. *There are two Numbers, if the Square of the greater is divided by the lesser, to this Quotient adding the greater, from which Sum subtracting the Square of the lesser, the Remainder is 100:*

*And the Sum of the two Numbers is 50. What are the Numbers sought?*

Let  $a$  = the greater,  $e$  = the lesser Number,  $m = 100$ ,  $p = 50$ .

$$\left\{ \begin{array}{l} 1 \mid \frac{aa}{e} + a - ee = m \\ 2 \mid a + e = p \end{array} \right\} \text{By the Question.}$$

In the first Equation both the unknown Quantities are to the first and second Power; but in the second Equation they are only to the first Power; therefore, according to the Directions, find the Value of  $a$  or  $e$ , in the second Equation.

$$2 - e \mid 3 \mid a = p - e$$

Because  $a$  is to the second Power in the first Equation, raise the third Equation to the second Power.

$$3 \odot 2 \mid 4 \mid aa = pp - 2pe + ee$$

Now for  $aa$  and  $a$  in the first Equation, write their respective Values,  $pp - 2pe + ee$ , and  $p - e$ , found by the third and fourth Equations, then we have,

1 . 3 . 4	5	$\frac{pp - 2pe + ee}{e} + p - e - ee = m,$ an Equation from which $a$ is exter- minated, and contains only the un- known Quantity $e$ .
5 $\times e$	6	$pp - 2pe + ee + pe - ee - eee = me$
That is	7	$pp - pe - eee = me$
7 + eee	8	$pp - pe = me + eee$
8 + pe	9	$pp = eee + me + pe$
9 in Numbers	10	$eee + 100e + 50e = 2500$
That is	11	$eee + 150e = 2500$

Here the Equation appears to be *adfected*, and to resolve it, let us suppose  $e = 9$ .

$$\text{Then } eee = 729$$

$$\text{And } 150e = 1350$$

2079 which being less than 2500, therefore  $e$  must be more than 9.

Then let  $r = 9$ , and  $y =$  what 9 wants of the true Value of  $e$ , then by *Case 1*, Art. 67, we have,

$$1 \odot 3 \mid \begin{array}{l} 1 \\ 2 \end{array} \mid \begin{array}{l} r + y = e \\ rrr + 3rry + 3ryy = eee, \text{ rejecting} \\ \text{the Powers of } y \text{ above } yy. \end{array}$$

Because in the given Equation  $e$  is multiplied by 150, therefore multiply the first Step by 150.

$$1 \times 150 \mid 3 \mid 150r + 150y = 150e$$

Now add the second and third Equations together, because the like Powers of  $e$  in the *adfected* Equation, are connected by the Sign +.

$$\begin{array}{l} 2 + 3 \\ \text{But} \end{array} \mid \begin{array}{l} 4 \\ 5 \end{array} \mid \begin{array}{l} rrr + 3rry + 3ryy + 150r + 150y \\ = eee + 150e \\ eee + 150e = 2500, \text{ by the given} \\ \text{Equation.} \end{array}$$

4 . 5	6	$rrr + 3rry + 3ryy + 150r + 150y$ $= 2500$
6 in Numbers	7	$729 + 243y + 27yy + 1350 + 150y$ $= 2500$
7 contracted	8	$2079 + 393y + 27yy = 2500$
8 — 2079	9	$393y + 27yy = 421$ Dividing by the Co-efficient of $yy$ .
9 ÷ 27	10	$14.55y + yy = 15.59$ Now dividing by the Co-efficient of $y$ <i>plus y.</i>
10 ÷ 14.55 + y	11	$y = \frac{15.59}{14.55 + y}$

Operation 14.55) 15.59 (1. =  $y$

I.

Divisor  $\overline{15.55}$

4 Remainder neglected.

$$r = 9$$

$$y = 1$$

$r + y = 10 = e$ , which being involved and tried will be found to be the true Root: Hence 10 is the lesser Number sought.

Then by the third Step of the Work to the Question  $a = p - e = 40$ , the greater Number sought.

In the Division for finding  $y$ , the Learner may observe, that as the two next Figures in the Quotient will be *Cyphers*, and in the Places of Fractions, and the third Figure being of so small a Value, I proceed no further in the Division, but leave it as in the Work, and so happen to find the true Value of  $e$ .

Question 85. Two Men, A and B, have such a Number of Pounds, that the Pounds A has, divided by the Pounds B has, and from this Quotient subtracting three times the Square of B's Pounds, and to the Remainder adding the Square of A's Pounds, the Sum is 27:

But if from the Pounds A has, there is subtracted the Pounds B has, the Remainder is 5. How many Pounds had each Man?

Put  $a$  = the Money of A,  $e$  = the Money of B,  $d = 27$ ,  $x = 5$ .

$$\left. \begin{array}{l|l} 1 & \frac{a}{e} - 3ee + aa = d \\ 2 & a - e = x \end{array} \right\} \text{By the Question.}$$

In

In the first Equation  $a$  and  $e$  being to the first and second Power, and to the first Power only in the second Equation, therefore by the Directions find the Value of  $a$ , or  $e$ , in the second Equation, suppose we find the Value of  $a$ .

$$2 + e \mid 3 \mid a = x + e$$

Raise this to the second Power, because  $a$  is to the second Power in the first Equation.

$$3 \oplus 2 \mid 4 \mid aa = xx + 2xe + ee$$

Now for  $a$  and  $aa$  in the first Equation, write their respective Values,  $x + e$ , and  $xx + 2xe + ee$ .

1 . 3 . 4	5	$\frac{x+e}{e} - 3ee + xx + 2xe + ee = d$ , here
		$a$ is exterminated, for the Equation contains only the unknown Quantity $e$ .
5 $\times e$	6	$x+e - 3eee + xxe + 2xe + ee = de$
That is	7	$x+e - 2eee + xxe + 2xe = de$
7 in Numbers	8	$5+e - 2eee + 25e + 10ee = 27e$
8 + 2eee	9	$5+e + 25e + 10ee = 2eee + 27e$
6 - 10ee	10	$5+e + 25e = 2eee - 10ee + 27e$
10 - e	11	$5 + 25e = 2eee - 10ee + 26e$
11 - 25e	12	$2eee - 10ee + e = 5$

To resolve this Equation, suppose  $e = 6$ .

$$\begin{array}{r}
 \text{Then } 2eee = 432 \\
 - 10ee = - 360 \\
 \hline
 72 \\
 + e = 6 \\
 \hline
 \end{array}$$

78 which being greater than 5, the Number in the given Equation, hence  $e$  cannot be so much as 6, therefore,

Let  $r = 6$ , and  $y =$  what 6 is too much, then by *Cafe* 2, Page 235.

$$1 \oplus 3 \mid \begin{array}{l} 1 \\ 2 \end{array} \mid \begin{array}{l} r - y = e \\ rrr - 3rry + 3ryy = eee \text{ re-} \\ \text{jecting the Powers of } y \text{ above } yy. \end{array}$$

Because



Because in the given Equation  $eee$  is multiplied by 2, therefore multiply the last Equation by 2.

$$2 \times 2 \mid 3 \mid 2rrr - 6rry + 6ryy = 2eee$$

Now raise  $r - y = e$  to the second Power, after which multiply it by 10, because it is 10  $ee$  in the given affected Equation.

$$\begin{array}{l} 1 \oplus 2 \mid 4 \mid rr - 2ry + yy = ee \\ 4 \times 10 \mid 5 \mid 10rr - 20ry + 10yy = 10ee \end{array}$$

Then add or subtract the Equations that are equal to 2  $eee$ , 10  $ee$  and  $e$ , according as those Quantities have the Signs + or —, in the given affected Equation.

$$\begin{array}{ll} 3 - 5 + 1 \mid 6 \mid 2rrr - 6rry + 6ryy - 10rr + 20ry - 10yy + r - y = 2eee - 10ee + e \\ \text{But} \mid 7 \mid 2eee - 10ee + e = 5, \text{ by the given Equation.} \\ 6 \cdot 7 \mid 8 \mid 2rrr - 6rry + 6ryy - 10rr + 20ry - 10yy + r - y = 5 \\ 8 \text{ in Numbers} \mid 9 \mid 432 - 216y + 36yy - 360 + 120y - 10yy + 6 - y = 5 \\ 9 \text{ contracted} \mid 10 \mid 78 - 97y + 26yy = 5 \end{array}$$

Transpose 5 it being less than 78

$$\begin{array}{ll} 10 - 5 \mid 11 \mid 73 - 97y + 26yy = 0, \text{ for one Side of the Equation subtracted from the other, the Remainder must be nothing, both Sides of the Equation being equal to one another.} \\ 11 + 97y \mid 12 \mid 73 + 26yy = 97y \\ 12 - 26yy \mid 13 \mid 97y - 26yy = 73 \\ \text{Divide by the Co-efficient of } yy. \\ 13 \div 26 \mid 14 \mid 3.73y - yy = 2.807. \\ \text{Now divide by the Co-efficient of } y \text{ minus } y. \\ 14 \div 3.73 - y \mid 15 \mid y = \frac{2.807}{3.73 - y} \\ \text{Operation } 3.73) \quad 2.807 \quad (1. = y \\ \quad \quad \quad - 1. \\ \quad \quad \quad \hline \quad \quad \quad 2.73 \\ \text{Divisor } 2.73 \hline \end{array}$$

7 Remainder neglected.

$$r = 6$$

$r = 6$  by Supposition,

$$-y = -1$$

$r - y = 5 = e$ , which being involved and tried it will be found to be the true Root, hence B had 5 Pounds.

Then by the third Step of the Work to the Question  $a = x + e = 10$  Pounds, the Money A had.

70. *The Numerical Method of resolving adaffected Equations being explained, we shall now show the Learner, that every adaffected Equation has as many Roots, either real or imaginary, as are the highest Dimensions of its unknown Quantity.*

For in any Equation where the highest Power of the unknown Quantity is the Biquadratic, or fourth Power, then there may be four Values of the unknown Quantity; if it is only to the third Power, then there may be three Values of the unknown Quantity, and so on: But there cannot be more Roots or Values of the unknown Quantity than there are Dimensions in the Equation.

These Roots are sometimes *affirmative*, and sometimes *negative*, and some Roots are *impossible*. The Reader observing how Quadratic Equations were compounded and generated, may better understand the Nature of these Roots. Thus,

Suppose  $a = 1$ , then  $a - 1 = 0$ , again suppose  $a = 2$ , then  $a - 2 = 0$ .

Now multiply these two together

$$\begin{array}{r} a - 1 = 0 \\ a - 2 = 0 \\ \hline aa - a = 0 \\ -2a + 2 = 0 \end{array}$$

An Equation of two Dimensions, which has two Roots, viz. 1 and 2

Again, let  $a = 3$ , then

$$\begin{array}{r} a - 3 = 0 \\ \hline aaa - 3aa + 2a = 0 \\ -3aa + 9a - 6 = 0 \end{array}$$

From multiplying these together, we have an Equation of three Dimensions, and which has 3 Roots, viz. 1 . 2 and 3.

Lastly, suppose  $a = -5$ , then

$$\begin{array}{r} a + 5 = 0 \\ \hline aaaa - 6aaa + 11aa - 6a = 0 \\ + 5aaa - 30aa + 55a - 30 = 0 \end{array}$$

An Equation of 4 Dimensions, and which has 4 Roots, viz. 1 . 2 . 3 . and -5, and so of any other Power.

These several Multiplications must all be  $= 0$  because the Multiplicand and Multiplier are each  $= 0$ .

By the same Method that we found the two Roots in Quadratic Equations, we may find the Roots of these Equations. For suppose we had this Equation  $aaaa - aaa - 19aa + 49a - 30 = 0$  given, which being resolved by the Method of *Converging Series*, we shall find  $a = 1$ , whence 1 is one of the Roots of the given adaffected Equation; now transpose 1 to make it  $a - 1 = 0$ , take the given Equation, which being equal to *nothing*, and dividing it by  $a - 1$ , the Quotient must be equal to *nothing*, thus,

$$\begin{array}{r}
 a-1=0) \quad aaaa - aaa - 19aa + 49a - 30 = 0 \quad (aaa - 19a + 30 = 0 \\
 \underline{aaaa - aaa} \\
 \phantom{a-1=0)} - 19aa + 49a - 30 \\
 \phantom{a-1=0)} \underline{- 19aa + 19a} \\
 \phantom{a-1=0)} \phantom{a-1=0)} 30a - 30 \\
 \phantom{a-1=0)} \phantom{a-1=0)} \underline{30a - 30} \\
 \phantom{a-1=0)} \phantom{a-1=0)} \phantom{a-1=0)} 0
 \end{array}$$

Here we find the Quotient to be  $aaa - 19a + 30 = 0$ , and solving this Equation by the Method of *Converging Series*, we shall find  $a = 3$ , for another of the Roots of the given adaffected Equation.

$$\begin{array}{r}
 \text{Then } a-3=0) \quad aaaa - 19a + 30 = 0 \quad (aa + 3a - 10 = 0 \\
 \phantom{a-3=0)} \underline{aaa - 3aa} \\
 \phantom{a-3=0)} \phantom{a-3=0)} 3aa - 19a + 30 \\
 \phantom{a-3=0)} \phantom{a-3=0)} \underline{3aa - 9a} \\
 \phantom{a-3=0)} \phantom{a-3=0)} \phantom{a-3=0)} - 10a + 30 \\
 \phantom{a-3=0)} \phantom{a-3=0)} \phantom{a-3=0)} \underline{- 10a + 30} \\
 \phantom{a-3=0)} \phantom{a-3=0)} \phantom{a-3=0)} \phantom{a-3=0)} 0
 \end{array}$$

Hence we have got this Quadratic Equation  $aa + 3a - 10 = 0$ , whence  $aa + 3a = 10$ , the two Roots of which are 2 and  $-5$ , the two remaining Roots of the given adaffected Equation; in the same Manner all the possible Roots of any other Equation are determined.

And to give the Learner an Instance where some of the Roots of an Equation are impossible :

Suppose  $aaa - 4aa + 4a - 16 = 0$ , by transposing 16 and resolving the Equation by the Method of *Converging Series*, we shall find  $a = 4$  : Then transposing 4 to make it  $a - 4 = 0$ , and making the given Equation equal to *nothing*, and dividing thus,

$$\begin{array}{r}
 a - 4 = 0 \quad a a a - 4 a a + 4 a - 16 = 0 \quad (a a + 4 = 0 \\
 \underline{a a a - 4 a a} \qquad \qquad \qquad 4 a - 16 \\
 \qquad \qquad \qquad \underline{4 a - 16} \\
 \qquad \qquad \qquad \qquad \qquad \qquad 0
 \end{array}$$

Because the Dividend and Divisor are both equal to *nothing*, therefore the Quotient must be equal to *nothing*; but if  $a a + 4 = 0$ , then  $a a = -4$  an Equation which has no real or *possible* Root in Nature, it being impossible to generate or produce a *negative* Square, for *minus* multiplied into *minus*, as well as *plus* multiplied into *plus*, makes the Product *affirmative*, or *plus*.

Question 86. Three Merchants, A, B, and C, found the Pounds A and B had gained, were equal to twice the Pounds C had gained:

But if the Pounds A gained were added to twice the Pounds B gained, and this Sum added to the Pounds C gained, it made 19 Pounds:

And the Sum of the Squares of each Person's Gain was equal to 77 Pounds. How much did each Person gain?

Let  $a$  = the Gain of A,  $e$  = the Gain of B,  $y$  = the Gain of C,  $m = 19$ ,  $p = 77$ .

$$\begin{array}{l|l|l}
 1 & 1 & a + e = 2y \\
 2 & 2 & a + 2e + y = m \\
 3 & 3 & a a + e e + y y = p \\
 1 - e & 4 & a = 2y - e. \text{ Raise this Equation to} \\
 & & \text{the second Power, it being } a a \text{ at} \\
 & & \text{the third Equation.} \\
 4 \ominus 2 & 5 & a a = 4y y - 4y e + e e
 \end{array}
 \left. \vphantom{\begin{array}{l} 1 \\ 2 \\ 3 \end{array}} \right\} \begin{array}{l} \text{By the} \\ \text{Question.} \end{array}$$

Now for  $a$ , and  $a a$  in the second and third Equations write their respective Values, *viz.*  $2y - e$ , and  $4y y - 4y e + e e$ .

$$\begin{array}{l|l|l}
 2 \cdot 4 & 6 & 3y + e = m \\
 3 \cdot 5 & 7 & 5y y - 4y e + 2e e = p.
 \end{array}$$

Here the Question is reduced to two Equations and two unknown Quantities, for  $a$  is exterminated, therefore in the sixth Equation, find the Value of  $e$ , or  $y$ , and raise it to the second Power, for those Quantities are to the second Power in the seventh Equation.

$6 - 3y$	$8$	$e = m - 3y$	
$8 \ominus 2$	$9$	$ee = mm - 6my + 9yy$	Multiply this
			Equation by 2, because it is $2ee$ in
			the seventh Equation.
$9 \times 2$	$10$	$2ee = 2mm - 12my + 18yy$	

Now in the seventh Equation for  $e$  and  $2ee$  write their Values at the eighth and tenth Steps.

$7 \cdot 8 \cdot 10$	$11$	$5yy - 4ym + 12yy + 2mm - 12my + 18yy = p$ , an Equation with only the unknown Quantity $y$ .
$11$ contracted	$12$	$35yy - 16my + 2mm = p$
$12 - 2mm$	$13$	$35yy - 16my = p - 2mm$ , here the Equation appears <i>quadratic</i> , and it is likewise <i>ambiguous</i> , for $2mm$ is greater than $p$ .
$13 \div 35$	$14$	$yy - \frac{16my}{35} = \frac{p - 2mm}{35}$
$14 \epsilon \square$	$15$	$yy - \frac{16my}{35} + \frac{256mm}{4900} = \frac{256mm}{4900} + \frac{p - 2mm}{35}$

The Co-efficient of  $y$  is  $\frac{16m}{35}$ , which being divided by 2, or  $\frac{2}{1}$  by the Rule in common Arithmetic for Division of Vulgar Fractions, the Quotient is  $\frac{16m}{70}$ , the Square of which is  $\frac{256mm}{4900}$ .

$15 \text{ } \text{uu } 2$	$16$	$y - \frac{16m}{70} = \sqrt{\frac{256mm}{4900} + \frac{p - 2mm}{35}}$
$16 + \frac{16m}{70}$	$17$	$y = \frac{16m}{70} \pm \sqrt{\frac{256mm}{4900} + \frac{p - mm}{35}}$
		$= 4.9999$ or $3.6857$
		But $4.9999$ is the Number, the Answer being 5. then if $y = 5$
By the 8th Step	$18$	$e = m - 3y = 4$
By the 4th Step	$19$	$a = 2y - e = 6$

Question 87. A, B, and C, having been at the Gaming-Table, found the Pounds A lost added to the Pounds C lost was equal to twice the Pounds B lost :

But the Pounds A lost added to the Pounds B lost, and this added to twice the Pounds C lost, the Sum was 22 Pounds :

And the Product of what A and B lost, being added to three times the Product of what B and C lost, the Sum was 120 Pounds. How much did each lose ?

Let  $a$  = the Sum A lost,  $e$  = the Sum B lost,  $y$  = the Sum C lost,  $d = 22$ ,  $n = 120$ .

$$\begin{array}{r|l}
 1 & a + y = 2e \\
 2 & a + e + 2y = d \\
 3 & ae + 3ey = n \\
 \hline
 1 - y & 4 \quad a = 2e - y \\
 2 \cdot 4 & 5 \quad 3e + y = d \\
 3 \cdot 4 & 6 \quad 2ee - ey + 3ey = n
 \end{array} \left. \vphantom{\begin{array}{l} 1 \\ 2 \\ 3 \end{array}} \right\} \text{By the Question.}$$

By the fifth and sixth Steps, the Question is reduced to two Equations, and two unknown Quantities, and because  $y$  is only to the first Power in both Equations, find the Value of  $y$  in each of them.

$$\begin{array}{r|l}
 5 - 3e & 7 \quad y = d - 3e \\
 6 - 2ee & 8 \quad 2ey = n - 2ee \\
 8 \div 2e & 9 \quad y = \frac{n - 2ee}{2e} \\
 \hline
 7 \cdot 9 & 10 \quad \frac{n - 2ee}{2e} = d - 3e \\
 10 \times 2e & 11 \quad n - 2ee = 2de - 6ee \\
 11 + 6ee & 12 \quad 4ee + n = 2de \\
 12 - n & 13 \quad 4ee - 2de = -n \\
 13 - 2de & 14 \quad \text{Here the Equation is quadratic and} \\
 & \quad \text{ambiguous.} \\
 & \quad \frac{de}{2} = -\frac{n}{4} \\
 14 \div 4 & 15 \quad ee - \frac{de}{2} + \frac{dd}{16} = \frac{dd}{16} - \frac{n}{4} \text{ for the} \\
 15 e \square & 16 \quad \text{Co-efficient of } e \text{ is } \frac{d}{2}, \text{ which being} \\
 & \quad \text{divided by 2 as in the last Question, the}
 \end{array}$$

		the Quotient is $\frac{d}{4}$ , the Square of
		which is $\frac{dd}{16}$ .
16 $\div$ 2	17	$e - \frac{d}{4} = \sqrt{\frac{dd}{16} - \frac{n}{4}}$
17 $+$ $\frac{d}{4}$	18	$e = \frac{d}{4} \pm \sqrt{\frac{dd}{16} - \frac{n}{4}} = 5.5 \pm .5$ (= 6, or 5, if $e = 6$ ,
Then by Step 7th	19	$y = d - 3e = 4$
And by Step 4th	20	$a = 2e - y = 8$

But if  $e = 5$ , then by the seventh Step  $y = d - 3e = 7$ ,  
and by the fourth Step  $a = 2e - y = 3$ .

Question 88. *There are two Numbers, the Sum of their Squares  
being added to their Sum, is 338 :*

*And their Product is 156. What are the Numbers ?*

Let  $a$  and  $e$  be the two Numbers sought,  $b = 338$ ,  $m = 156$ .

Then	1	$aa + ee + a + e = b$	} By the Question.
	2	$ae = m$	
2 $\div$ e	3	$a = \frac{m}{e}$	
3 $\odot$ 2	4	$aa = \frac{mm}{ee}$	
1 $\cdot$ 3 $\cdot$ 4	5	$\frac{mm}{ee} + ee + \frac{m}{e} + e = b$ , an Equation having the unknown Quantity $e$ only.	
5 $\times$ ee	6	$mm + eeee + \frac{eem}{e} + eee = bee$	
		But as $\frac{eem}{e} = em$ , the $e$ being rejected (by Art. 20.	
Hence	7	$mm + eeee + em + eee = bee$	

There being only the known Quantity  $mm$ , transpose the  
others so that  $mm$  may be at last affirmative ; and this is to be  
observed,

observed, that in transposing the Quantities in these *advised* Equations, the Side of the Equation which is known may at last be *affirmative*.

7 — eeee	8	$mm + em + eee = bee - eeee$
8 — eee	9	$mm + em = bee - eeee - eee$
9 — em	10	$mm = bee - eeee - eee - me$
Or	11	$- eeee - eee + bee - me = mm,$ it being the common Method to place these Equations, according to the highest Power of the unknown Quantity.
11 in Numbers	12	$- eeee - eee + 338 ee - 156 e = 24336.$

Now suppose  $e = 10$ .

$$\begin{array}{rcl}
 \text{Then } - eeee & = & - 10000 \\
 - eee & = & - 1000 \\
 & & \hline
 & & - 11000 \\
 + 338 ee & = & 33800 \\
 & & \hline
 & & 22800 \\
 - 156 e & = & - 1560 \\
 & & \hline
 - eeee - eee + 338 ee - 156 e & = & 21240 \text{ which being less than} \\
 24336 \text{ the Number in the given Equation, therefore } e & \text{must be} & \\
 \text{more than } 10. & & 
 \end{array}$$

Let  $r = 10$ , and put  $y =$  what it wants of being the true Root.

Then	1	$r + y = e$
1 ⊙ 4	2	$rrrr + 4rrry + 6rryy = eeee,$ all the Powers of $y$ above $yy$ being rejected.
1 ⊙ 3	3	$rrr + 3rry + 3ryy = eee,$ rejecting all the Powers of $y$ above $yy$ .
1 ⊙ 2	4	$rr + 2ry + yy = ee$

Because in the given Equation it is  $338 ee$ , therefore multiply the fourth Equation by 338.

$$4 \times 338 \mid 5 \mid 338 rr + 676 ry + 338 yy = 338 ee$$

Because



Because in the given Equation it is  $156e$ , therefore multiply the first Equation by  $156$ .

$$1 \quad 156 \mid 6 \mid 156r + 156y = 156e$$

Now the second, third, fifth, and sixth Equations being equal to the several Powers of  $e$ , and multiplied by the same Coefficients as in the given Equation, add or subtract them according to the Signs those Powers have in that Equation.

$-2 - 3 + 5 - 6$	$7$	$-rrrr - 4rrry - 6rryy - rrr - 3rry$ $- 3ryy + 338rr + 676ry + 338yy$ $- 156r - 156y = -eeee - eee$ $+ 338ee - 156e$
But	$8$	$-eeee - eee + 338ee - 156e = 24336$ by the given Equation.
$7 \cdot 8$	$9$	$-rrrr - 4rrry - 6rryy - rrr - 3rry$ $- 3ryy + 338rr + 676ry + 338yy$ $- 156r - 156y = 24336$
$9$ in Numbers	$10$	$-10000 - 4000y - 600yy - 1000$ $- 300y - 30yy + 33800 + 6760y$ $+ 338yy - 1560 - 156y = 24336$
$10$ contracted	$11$	$21240 + 2304y - 292yy = 24336$
$11 - 21240$	$12$	$2304y - 292yy = 3096$
$12 \div 292$	$13$	Now divide by the Co-efficient of $yy$ . $7.89y - yy = 10.6$ And dividing by the Co-efficient of $y$ , minus $y$ ,
$13 \div 7.89 - y$	$14$	$y = \frac{10.6}{7.89 - y}$

Operation  $7.89 \mid 10.60 \mid 1.7 = y$

— 1.

$$\text{Divisor } \begin{array}{r} 6.89 \\ - 1.7 \\ \hline 5.19 \end{array}$$

$$\begin{array}{r} 6.89 \\ - 1.7 \\ \hline 3710 \\ 3633 \\ \hline 77 \end{array}$$

$$\text{Divisor } \begin{array}{r} 5.19 \\ - 1.7 \\ \hline 3710 \\ 3633 \\ \hline 77 \end{array}$$

$r = 10$  by Supposition.

$$+ y = 1.7$$

$r + y = 11.7$  which being involved and tried, it will be found too little ; therefore for a second Operation,

Suppose  $r = 11.7$  and  $y$  what it wants of the true Root.

Then

Then	1	$r + y = e$
$1 \ominus 4$	2	$rrrr + 4rrry + 6rryy = eeee$ , the Powers of $y$ above $yy$ being rejected.
$1 \ominus 3$	3	$rrr + 3rry + 3ryy = eee$ , the Powers of $y$ above $yy$ being rejected.
$1 \ominus 2$	4	$rr + 2ry + yy = ee$

Because in the given Equation it is  $338 ee$ , therefore multiply the last Equation by  $338$ .

$$4 \times 338 \mid 5 \mid 338 rr + 676 ry + 338 yy = 338 ee$$

Because in the given Equation it is  $156 e$ , therefore multiply the first Equation by  $156$ .

$$1 \times 156 \mid 6 \mid 156 r + 156 y = 156 e$$

Now add or subtract the Equations that are equal to  $eeee$ ,  $eee$ ,  $338 ee$  and  $156 e$ , according to the Signs those Quantities have in the given *affected* Equation.

$-2 - 3 + 5 - 6$	7	$-rrrr - 4rrry - 6rryy - rrr - 3rry$ $- 3ryy + 338 rr + 676 ry + 338 yy$ $- 156 r - 156 y = -eeee - eee$ $+ 338 ee - 156 e$
But	8	$-eeee - eee + 338 ee - 156 e = 24336$ , by the given Equation.
$7 \cdot 8$	9	$-rrrr - 4rrry - 6rryy - rrr - 3rry$ $- 3ryy + 338 rr + 676 ry + 338 yy$ $- 156 r - 156 y = 24336$
9 in Numbers	10	$-18738.8721 - 6406.452y - 821.34yy$ $- 1601.613 - 410.67y - 35.1yy$ $+ 46268.82 + 7909.2y + 338yy$ $- 1825.2 - 156y = 24336$
10 contracted	11	$24103.1349 + 936.078y - 518.44yy$ $= 24336$
11 —	12	$936.078y - 518.44yy = 232.8651$ Dividing by the Co-efficient of $yy$ .
12 ÷	13	$1.805y - yy = .4491$ . Now dividing by the Co-efficient of $y$ <i>minus y</i> , that is, by $1.805 - y$ .
$13 \div 1.805 - y$	14	$y = \frac{.4491}{1.805 - y}$

Operation 1.805) .4491 (.297 =  $\gamma$ .

$$\begin{array}{r}
 \text{Divisor } 1.805 \quad 3210 \\
 \text{---} \quad .29 \quad \text{---} \\
 \text{Divisor } 1.315 \quad 12810 \\
 \text{---} \quad 97 \quad \text{---} \\
 \text{Divisor } 1.218 \quad 9750 \\
 \quad \quad \quad 8526 \\
 \quad \quad \quad \text{---} \\
 \quad \quad \quad 1224
 \end{array}$$

$r = 11.7$  by Supposition,

$$+ y = .297$$

$r + y$  is 11.997 =  $e$ , which is something too little, the true Value being 12. but this may inform the Learner of the Nature of solving these high *adfect*ed Equations, every Operation approaching nearer and nearer to the true Root, from whence it may be found to any assignable Degree of Exactness.

And having found  $e$  to be 12, then by the third Step of the Work to the Question, we have  $a = \frac{m}{e} = 13$ , the other Number sought.

71. *The Method of resolving Equations when the unknown Quantity is to several Powers in both Equations.*

When both the unknown Quantities are to the first and second Power in both Equations, find the Value of the Square of the unknown Quantity in each Equation, and make these two Equations equal to one another; which Equation will have the first Power only of the unknown Quantity, its Square being exterminated by that Equation.

Then find the Value of the first Power of the unknown Quantity in this last Equation, which raise to the second Power; and in either of the two given Equations in which it may be most conveniently done, for this unknown Quantity and its several Powers, write their respective Values, which will give an Equation with only one unknown Quantity, and is to be reduced by the Rules already explained.

Question 89. To find two Numbers, the Sum of whose Squares is equal to the lesser multiplied by 20 :

And the Square of the lesser being added to their Product, the Sum is 16.

Let  $a$  = the greater Number,  $e$  = the lesser Number,  
 $m = 20$ ,  $d = 16$ .

1	$aa + ee = me$	} By the Question.
2	$ee + ae = d$	
Begin to exterminate $ee$ according to the Directions, that is, find the Value of $ee$ in both the given Equations.		
1 — $aa$	3	$ee = me - aa$
2 — $ae$	4	$ee = d - ae$
3 · 4	5	$me - aa = d - ae$ , here $ee$ is exterminated, now find the Value of $e$ .
5 + $ae$	6	$me + ae - aa = d$
6 + $aa$	7	$me + ae = d + aa$
7 ÷ $m + a$	8	$e = \frac{d + aa}{m + a}$
Raife this Value of $e$ to the second Power.		
8 ⊙ 2	9	$ee = \frac{dd + 2daa + aaaa}{mm + 2ma + aa}$

Now in the first Equation for  $ee$  and  $e$ , write their respective Values at the eighth and ninth Steps.

1 · 9 · 8	10	$aa + \frac{dd + 2daa + aaaa}{mm + 2ma + aa} =$ $\frac{md + maa}{m + a}$ an Equation clear of $e$ , having only the unknown Quantity $a$ .
-----------	----	--

To clear this Equation of the Fractions, observe that  $mm + 2ma + aa$  is the Square of  $m + a$ , the former arising from the Involution of the latter by the eighth and ninth Steps, and in the Multiplication of Fractions, it being the same thing to divide the Divisor, as to multiply the Dividend, to multiply  $\frac{dd + 2daa + aaaa}{mm + 2ma + aa}$ , by  $m + a$ , we only change the Divisor to  $m + a$ , that being the Quotient of  $mm + 2ma + aa$  divided by  $m + a$ , the rest of the Multiplication is the same as usual.

$10 \times m + a$	11	$m a a + a a a + \frac{d d + 2 d a a + a a a a}{m + a}$
		$= m d + m a a$
$11 \times m + a$	12	$m m a a + m a a a + m a a a + a a a a + d d$ $+ 2 d a a + a a a a = m m d + m m a a$ $+ m d a + m a a a$
$12 - m a a a$	13	$m m a a + m a a a + a a a a + d d + 2 d a a$ $+ a a a a = m m d + m m a a + m d a$
$14 - m m a a$	14	$m a a a + a a a a + d d + 2 d a a + a a a a$ $= m m d + m d a$
$14 - m d a$	15	$m a a a + 2 a a a a + d d + 2 d a a -$ $m d a = m m d$
$15 - d d$	16	$2 a a a a + m a a a + 2 d a a - m d a$ $= m m d - d d$
16 in Numbers	17	$2 a a a a + 20 a a a + 32 a a - 320 a = 6144$ Because the Co-efficient of $a a a a$ will divide the other Co-efficients without any Remainder, divide by it.
$17 \div 2$	18	$a a a a + 10 a a a + 16 a a - 160 a = 3072$

Which Equation being resolved by the Method of *Converging Series*, we shall find  $a = 6$ , or nearly to it, 6 being the true Root, from whence by the eighth Step  $e = 2$ .

Question 90. *There are two Numbers, if the greater is added to its Square, and from this Sum we subtract the Square of the lesser, the Remainder is 94:*

*But the Square of the lesser, being added to the lesser, this Sum is equal to twice the greater.*

Let  $a =$  the greater Number,  $e =$  the lesser Number,  $m = 94$ .

	1	$a + a a - e e = m$	} By the Question.
	2	$e e + e = 2 a$	
		Begin with finding the Value of $e e$ in each Equation.	
$1 + e e$	3	$a a + a = m + e e$	
$3 - m$	4	$a a + a - m = e e$	
$2 - e$	5	$e e = 2 a - e$	
$4 \cdot 5$	6	$a a + a - m = 2 a - e$ , here $e e$ is ex-terminated, now find the Value of $e$ .	
$6 + e$	7	$e + a a + a - m = 2 a$	
$7 - a$	8	$e + a a - m = a$	
		L 1 2	$8 + m$

$8 + m$	9	$e + aa = a + m$
$9 - aa$	10	$e = a + m - aa$
$10 \odot 2$	11	Raise this Value of $e$ to the second Power. $ee = aa + 2am + mm - 2aaa - 2maa + aaaa$

Now for  $ee$  and  $e$  in the second Equation, write their respective Values, found at the tenth and eleventh Steps.

$2 \cdot 11 \cdot 10$	12	$aa + 2am + mm - 2aaa - 2maa + aaaa$
		$+ a + m - aa = 2a$
12 in Numbers	13	$188a + 8836 - 2aaa - 188aa + aaaa$
		$+ a + 94 = 2a$
13 contracted	14	$187a + 8930 - 2aaa - 188aa + aaaa = 0$
		Transpose the several Powers of $a$ , that 8930 the known Part of the Equation may be affirmative.
$14 - aaaa$	15	$-aaaa = 187a + 8930 - 2aaa - 188aa$
$15 + 2aaa$	16	$-aaaa + 2aaa = 187a + 8930 - 188aa$
$16 + 188aa$	17	$-aaaa + 2aaa + 188aa = 187a + 8930$
$17 - 187a$	18	$-aaaa + 2aaa + 188aa - 187a = 8930$

Which Equation being resolved, we shall find  $a = 10$ , or nearly to it, 10 being the true Root.

Then by the tenth Step  $e = a + m - aa = 4$ .

We shall now proceed to the Solution of several Geometrical Problems upon the same *general* Principles, and if the Learner is not sufficiently acquainted with the Elements of Geometry, to discover how the Equations are formed from the Properties of the Figure, he may omit these Questions, and proceed to the others which require no Knowledge in Geometry.

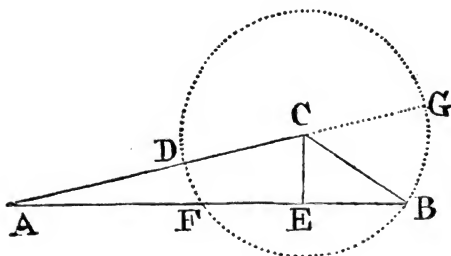
**Question 91.** *In the oblique Triangle ABC, given the Difference between the Sides AC and BC = 8, and the Difference between the Segments of the Base AE and EB = 10, and the Perpendicular CE, let fall from the vertical Angle C, upon the Base AB = 16. To find the Sides AC, CB, and Base AB?*

Upon C as a Center, with the Radius CB, draw the Circle GBFD, and continue AC to G.

Hence CB = CD, as Radius of the same Circle, whence AD is the Difference of the Sides, or the Difference between AC and CB = 8.

And

And  $BE = EF$ , as  $FB$  is bisected at  $E$  by the 3. e. 3 hence  $AF$  is the Difference of the Segments of the Base, or the Difference between  $AE$  and  $EB = 10$ .



Let  $AD = d = 8$ , and  $DC = CB = e$ , then  $AC = d + e$ .

Let  $AF = b = 10$ , and  $FE = EB = a$ , hence  $AB = b + 2a$ .

Let  $CE = p = 16$ .

Having two unknown Quantities,  $a$  and  $e$ , and no Equation from the Conditions of the Question, we must raise two Equations from the Properties of the Figure.

Now the Lines  $AG$  and  $AB$  are drawn from within a Circle, and touch without at the Point  $A$ , therefore by 37. e. 3  $AG \times AD = AB \times AF$ , all which Lines are expressed in Symbols, except  $AG$ , but  $CG = CD = e$ , and  $AC = d + e$ , therefore  $AG = d + 2e$ , hence we have in Symbols,

	1	$d + 2e \times d = b + 2a \times b$ the short Lines over the Quantities, signifies that they are both to be multiplied by the Quantity which follows the Sign of Multiplication.
		But the Triangle $CEB$ is right-angled, therefore by
47. e. 1	2	$pp + aa = ee$
From the first	3	$dd + 2de = bb + 2ba$
		Now find the Value of either $a$ or $e$ , in the third Equation.
3 — dd	4	$2de = bb + 2ba - dd$

But as we shall have Occasion to square this Equation, for when the Value of  $e$  is found, that Equation must be raised to the second

second Power, it being  $ee$  in the second Equation; and  $bb - dd$  being a known Quantity, to avoid Trouble, substitute  $x = bb - dd$ .

Then	5	$2de = x + 2ba$
$5 \div 2d$	6	$e = \frac{x + 2ba}{2d}$
		Raise this to the second Power, because it is $ee$ in the second Equation.
$6 \odot 2$	7	$ee = \frac{xx + 4xba + 4bbba}{4dd}$
$2 \cdot 7$	8	$\frac{xx + 4xba + 4bbba}{4dd} = pp + aa$ (an Equation clear of $e$ .)
$8 \times 4dd$	9	$xx + 4xba + 4bbba = 4ddpp + 4ddaa$
		Bring all the Powers of $a$ to one Side of the Equation.
$9 - 4ddaa$	10	$4bbba - 4ddaa + 4xba + xx = 4ddpp$
$10 - xx$	11	$4bbba - 4ddaa + 4xba = 4ddpp - xx$

Here the Equation appears *quadratic*, the unknown Quantity being only to the first and second Power; but as the Square of the unknown Quantity has Co-efficients, therefore by Article 58, divide the Equation by  $4bb - 4dd$ , the Co-efficients of  $aa$ .

$$11 \div 4bb - 4dd \quad 12 \quad aa + \frac{4xba}{4bb - 4dd} = \frac{4ddpp - xx}{4bb - 4dd}$$

The Work being now prepared for *compleating the Square*, and the Co-efficient of  $a$  being a Fraction, to save the Trouble of dividing it by 2, and squaring the Quotient according to Art. 57.

substitute  $y = \frac{4xb}{4bb - 4dd}$ ,

Then	13	$aa + ya = \frac{4ddpp - xx}{4bb - 4dd}$
$13 \square$	14	$aa + ya + \frac{yy}{4} = \frac{4ddpp - xx}{4bb - 4dd} + \frac{yy}{4}$
$14 \omega 2$	15	$a + \frac{y}{2} = \sqrt{\frac{4ddpp - xx}{4bb - 4dd} + \frac{yy}{4}}$
$15 - \frac{y}{2}$	16	$a = \sqrt{\frac{4ddpp - xx}{4bb - 4dd} + \frac{yy}{4}} - \frac{y}{2}$ $= 16.7.$

Then



Then by Step 6th

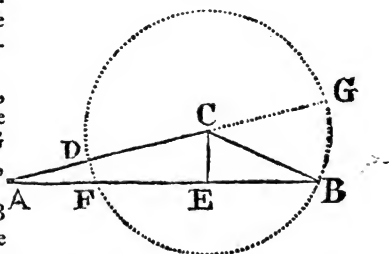
17		$e = \frac{x + 2ba}{2d} = 23.12$
Hence	18	$AC = d + e = 31.12$
	19	$CB = e = 23.12$
	20	$AB = b + 2a = 43.4$

Question 92. In the oblique Triangle ABC, there is given the Sum of the Sides AC and BC = 8, and the Difference of the Segments of the Base AE and BE = 2, with the Perpendicular CE, let fall from the vertical Angle at C upon the Base AB = 1. To find the Sides AC, BC, and Base AB?

Upon C as a Center with the Radius CB, draw the Circle GBFD, and continue AC to G.

Then CG = CB = CD, being all Radii of the same Circle, whence AG is the Sum of the Sides, or AC + CB = 8.

And FE = EB, for FB is bisected at E, by the 3. e. 3, whence AF is the Difference of the Segments of the Base, or the Difference between AE and BE = 2.



The Construction of this Figure being the same as the last, we can raise the same two Equations from the Figure, but instead of AD being given, we have AG given. Let AG, or AC + CB = s = 8, and DC = CG = a, whence DG = 2a, then AG - DG = AD = s - 2a.

Put AF = d = 2, and FE = EB = e, then AB = d + 2e, let CE = p = 1.

Now as in the last Question, because the Lines AG and AB are drawn from the Circumference within the Circle, and touch at the Point A without the Circle, hence by 37. e. 3 AG x AD = AB x AF, that is,

in Symbols		1		$s \times s - 2a = d + 2e \times d$
That is		2		$ss - 2sa = dd + 2de$
by 47. e. 1		3		$pp + ee = aa$ the Triangle CEB being right-angled, and DC = CB = a.

Here the Question is expressed by two Equations, and two unknown Quantities.

$2 + 2sa$	4	$ss = dd + 2de + 2sa$
$4 - dd$	5	$ss - dd = 2de + 2sa$
$5 - 2de$	6	$2sa = ss - dd - 2de$
		Substitute as before $x = ss - dd$ , for when the Value of $a$ is found, it must be raised to the second Power, to com- pare it with $aa$ in the third Equation.
Then	7	$2sa = x - 2de$
$7 \div 2s$	8	$a = \frac{x - 2de}{2s}$
		Raise this Equation to the second Power, because it is $aa$ in the third Equation.
$8 \odot 2$	9	$aa = \frac{xx - 4xde + 4ddee}{4ss}$
$3 \cdot 9$	10	$pp + ee = \frac{xx - 4xde + 4ddee}{4ss}$
$10 \times 4ss$	11	$4sspp + 4ssee = xx - 4xde + 4ddee$
		Now bring all the Powers of $e$ to one Side of the Equation.
$11 - 4ddee$	12	$4ssee - 4ddee + 4sspp = xx - 4xde$
$12 + 4xde$	13	$4ssee - 4ddee + 4xde + 4sspp = xx$
$13 - 4sspp$	14	$4ssee - 4ddee + 4xde = xx - 4sspp$

Here the Equation appears *quadratic*, but is not *ambiguous*, for  $xx$  is greater than  $4sspp$ . Dividing by the Co-efficient of  $ee$ , by Art. 58.

$$14 \div 4ss - 4dd \quad 15 \quad ee + \frac{4xde}{4ss - 4dd} = \frac{xx - 4sspp}{4ss - 4dd}$$

The Work being prepared for completing the Square, substitute  $y = \frac{4xd}{4ss - 4dd} = \frac{xd}{ss - dd}$ ,

Then	16	$ee + ye = \frac{xx - 4sspp}{4ss - 4dd}$
$16 \odot 4$	17	$ee + ye + \frac{yy}{4} = \frac{xx - 4sspp}{4ss - 4dd} + \frac{yy}{4}$

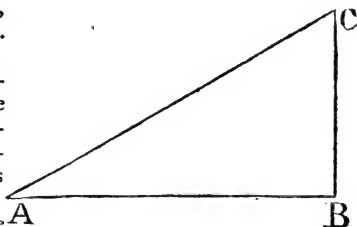
17	$w 2$	18	$e + \frac{y}{2} = \sqrt{\frac{xx - 4sspp}{4ss - 4dd}} + \frac{yy}{4}$
18	$-\frac{y}{2}$	19	$e = \sqrt{\frac{xx - 4sspp}{4ss - 4dd}} + \frac{yy}{4} : -\frac{y}{2}$ $= 2.86$
Then by Step 8th		20	$a = \frac{x - 2de}{2s} = 3.03 = BC.$
Hence		21	$AC = s - a = 4.97$
And		22	$BA = d + 2e = 7.72$

Question 93. In the right-angled Triangle ABC, there is given the Area of the Triangle equal to 24, and the Sum of the Hypotenuse AC and Perpendicular BC equal to 16. To find the Sides of the Triangle?

Let  $AC = y$ ,  $AB = a$ ,  
 $BC = e$ ,  $s = 24$ ,  $d = 16$ .

Here being three unknown Quantities, there must be raised three Equations from the Question, and the Properties of the Figure.

Now as the Triangle ABC is right-angled, therefore,



By 47 . e . i

1	$aa + ee = yy$	} By the Question.
2	$y + e = d$	
3	$\frac{ae}{2} = s$ , from the Rule for finding the Area of the Triangle, for the	

Product of the Base and Perpendicular of any Triangle being divided by 2, the Quotient is the Area.

The first Equation has all the three unknown Quantities, but the other two have only two of them. Now if we take that Quantity which is in all the three Equations, and find the Value of it in one of them, and in its Room write that Value in the other two Equations, the Question will be then reduced to two Equations, and two unknown Quantities; thus in the second Equation find the Value of  $e$ .

$2 - y$	4	$e = d - y$	} Two Equations with only two un- known Quantities.
		But as it is $ee$ in the first Equation, therefore,	
$4 \text{ } \odot \text{ } 2$	5	$ee = dd - 2dy + yy$	
$1 \cdot 5$	6	$aa + dd - 2dy + yy = yy$	
		$ad - ay = s$	
$3 \cdot 4$	7	$\frac{ad - ay}{2} = s$	
		Find the Value of $y$ , in each of these Equations.	
$6 - yy$	8	$aa + dd - 2dy = 0$ , for $yy$ being taken away by the Subtraction, that Side of the Equation is <i>nothing</i> .	
$9 + 2dy$	9	$2dy = aa + dd$	
$9 \div 2d$	10	$y = \frac{aa + dd}{2d}$	
$7 \times 2$	11	$ad - ay = 2s$	
$11 + ay$	12	$ad = 2s + ay$	
$12 - 2s$	13	$ay = ad - 2s$	
$13 \div a$	14	$y = \frac{ad - 2s}{a}$	
		$\frac{aa + dd}{2d} = \frac{ad - 2s}{a}$	
$10 \cdot 14$	15	$\frac{aa + dd}{2d} = \frac{ad - 2s}{a}$	
$15 \times a$	16	$\frac{aaa + dda}{2d} = ad - 2s$	
$16 \times 2d$	17	$aaa + dda = 2dda - 4ds$	
$17 \text{ in Numbers}$	18	$aaa + 256a = 512a - 1536$	
$18 - 512a$	19	$aaa - 256a = -1536$	

Here the Equation is *affected*, therefore transpose the Quantity so that the Side of the Equation which is known may have the *affirmative* Sign.

$19 + 256a$	20	$aaa = 256a - 1536$
$20 + 1536$	21	$aaa + 1536 = 256a$
$21 - aaa$	22	$-aaa + 256a = 1536$

Which Equation being resolved by the Method of *Converging Series*, we shall find  $a = 8$  nearly, for 8 is the true Root; from whence the other Sides of the Triangle are easily determined.

Question 94. In the right-angled Triangle ABC, there is drawn GE parallel to the Perpendicular BC, given the Perpendicular BC = 24, the Segment of the Hypotenuse EC = 15, and

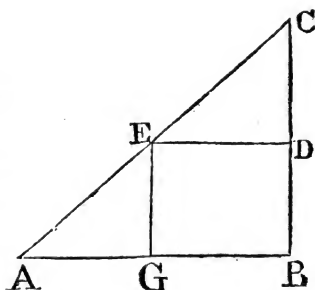
and the Segment of the Base  $AG = 20$ . To find the Hypotenuse  $AC$  and Base  $AB$ ?

Draw  $ED$  parallel to  $AB$ .

Let  $BC = c = 24$ ,  $EC = n = 15$ ,  $AG = b = 20$ ,  $GB = ED = a$ , then  $AB = b + a$ ,  $AE = e$ , then  $AC = n + e$ .

Here being two unknown Quantities, we must raise two Equations.

Now the Triangles  $AGE$  and  $EDC$  are similar, hence,

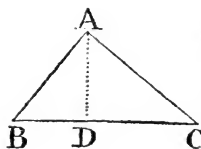


by 4 . e . 6	1	$AG : AE :: ED : EC$
in Symbols	2	$b : e :: a : n$
whence	3	$ae = bn$ , for Quantities that are in continual Proportion, the Product of the Extremes and Means are equal.
by 47 . e . I	4	$bb + 2ba + aa + cc = nn + 2ne + ee$ , the Triangle $ABC$ being right-angled, and as these two last Equations contain the Question, therefore
$3 \div e$	5	$a = \frac{bn}{e}$
$5 \odot 2$	6	$aa = \frac{bbnn}{ee}$
$4 \cdot 5 \cdot 6$	7	$bb + \frac{2bbn}{e} + \frac{bbnn}{ee} + cc = nn + 2ne (+ ee$
$7 \times e$	8	$bbe + 2bbn + \frac{bbnn}{e} + cce = nne (+ 2nee + eee$
$8 \times e$	9	$bbee + 2bbne + bbnn + ccee = nnee + 2neee + eeee$
9 in Numbers	10	$400ee + 12000e + 90000 + 576ee = 225ee + 30eee + eeee$
that is	11	$976ee + 12000e + 90000 = 225ee + 30eee + eeee$
$11 - 225ee$	12	$751ee + 12000e + 90000 = 30eee + eeee$
$12 - 751ee$	13	$eee + 30eee - 751ee = 12000e + 90000$
$13 - 12000e$	14	$eee + 30eee - 751ee - 12000e = 90000$

Which Equation being resolved by the Method of *Converging Series*, we shall find  $e = 25$  nearly, for 25 is the true Root.

Then by the fifth Step  $a = 12$ , whence  $AC = n + e = 40$ , and  $AB = b + a = 32$ .

Question 95. In the Triangle ABC, given the Base BC = 42.5 and the Angle at B =  $49^\circ : 45'$  and the Angle at C =  $42^\circ : 30'$  to find the Perpendicular AD, let fall from the vertical Angle at A upon the Base BC.



The Triangle ADB is right-angled, and the Angle ABD being given, all the Angles of the Triangle ABD are known, therefore by plain Trigonometry, we can find the Ratio between the Sides BD, and AD, though we do not know the Length of either of them, for as the Sine of the Angle BAD, is to the

Logarithm of any Number assumed for the Side BD, so is the Sine of the Angle at B to a fourth Number, which is the Logarithm of the proportional Number for the Side AD.

Therefore assuming *Unity*, or 1, for the Side BD, we have

As the Sine of the Angle at A	-	$40^\circ : 15'$	-	9.810316
Is to the Log. of the Side BD	-	1.	-	0.0
So is the Sine of the Angle at B	-	$49^\circ : 45'$	-	9.882657
				9.882657
				9.810316

To the Log. of the Side AD - 1.18 - - .072341

Hence the Sides AD and BD are to one another, as 1.18 is to 1.

Now let  $BC = b = 42.5$   $AD = a$ ,  $m = 1.18$  and  $p = 1$ . Consequently,

$$\left| 1 \right| m : p :: a : \frac{p a}{m} = BD, \text{ that is, as}$$

expresses the Proportion of AD and DB, are to one another, so is the true Length of AD to the true Length of BD.

By the same Reasoning, in the Triangle ADC, because all the Angles are known, therefore the Ratio of the Sides AD and DC are known, and assuming AD to be = 1. and proceeding by Trigonometry as before, we shall find the proportional Number for CD to be 1.1

Now

Now putting  $d = 1.1$  and  $p = 1$ , as before, we have

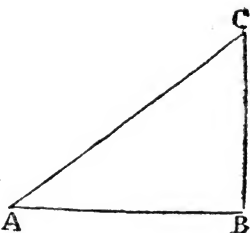
$$\left| \begin{array}{l} 2 \\ \hline \end{array} \right| p : d :: a : \frac{da}{p} = DC, \text{ by the same Reasoning as at the first Step.}$$

And as we have now expressed in Symbols the two Parts BD and DC of the Base BC,

$$\begin{array}{l} \text{Hence} \\ 3 \times m \\ 4 \times p \\ 5 \div p p + m b \end{array} \left| \begin{array}{l} 3 \\ 4 \\ 5 \\ 6 \\ \hline \end{array} \right| \begin{array}{l} \frac{pa}{m} + \frac{da}{p} = b, \text{ that is, } BD + DC = BC. \\ pa + \frac{mda}{p} = mb \\ ppa + mda = mbp \\ a = \frac{mbp}{pp + md} = 21.82 = AD. \end{array}$$

Question 96. In the right-angled Triangle ABC, there is given the Sum of the Sides equal to 12, and the Area equal to 6. To find the Hypothenufe AC?

Let  $s = 12$ ,  $b = 6$ ,  $BC = a$ ,  $AC = y$ .



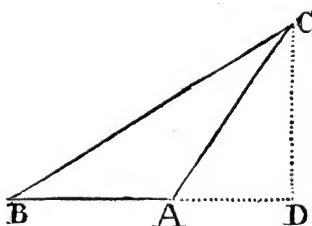
Then by the Property of the Triangle,  $AB = \sqrt{yy - aa}$ .

$$\begin{array}{l} \text{Hence} \\ 1 \\ 2 \end{array} \left| \begin{array}{l} 1 \\ 2 \\ \hline \end{array} \right| \begin{array}{l} a + y + \sqrt{yy - aa} = s \text{ by the (Question.)} \\ \frac{a}{2} \sqrt{yy - aa} = b, \text{ from the Rule for (finding the Area of the Triangle.)} \end{array}$$

Now because there is the same Surd in both Equations, find what the Surd is equal to in the second Equation, and write its Value for the Surd in the first Equation.

$$\begin{array}{l} 2 \div \frac{a}{2} \\ 3 \end{array} \left| \begin{array}{l} 3 \\ \hline \end{array} \right| \begin{array}{l} \sqrt{yy - aa} = \frac{2b}{a} \text{ for } b = \frac{b}{1} \text{ and } \frac{b}{1} \\ \div \frac{a}{2} = \frac{2b}{a}, \text{ by the Rule for Division of Fractions in common Arithmetic.} \end{array}$$

1 . 3	4	$a + y + \frac{2b}{a} = s$
4 x a	5	$aa + ay + 2b = sa$ , now the Question is contained in the first and fifth Equations.
1 —	6	$\sqrt{yy - aa} = s - a - y$
6 ⊖ 2	7	$yy - aa = ss - 2sa - 2sy + 2ay + aa + yy$
7 ±	8	$2aa = 2sa + 2sy - 2ay - ss$
5 —	9	$aa = sa - ay - 2b$
9 x 2	10	$2aa = 2sa - 2ay - 4b$
8 . 10	11	$2sa - 2ay - 4b = 2sa + 2sy - 2ay - ss$
11 + 2 ay	12	$2sa - 4b = 2sa + 2sy - ss$
12 — 2 sa	13	$-4b = 2sy - ss$
13 + ss	14	$2sy = ss - 4b$
13 ÷ 2 s	15	$y = \frac{ss - 4b}{2s} = 5 = A.C.$



Question 97. In the Triangle ABC, there is given the Sum of the Sides  $BC + BA + AC = 85$ , the Area = 200, and the Angle at A =  $124^\circ$ . To find the Sides of the Triangle?

Let  $s = 85$ ,  $b = 200$ ,  $AC = a$ , because the Angle  $BAC = 124^\circ$ , the Angle  $CAD = 56^\circ$ , and  $CD$  being a Perpendicular let fall on  $AB$  continued, all the Angles of the Triangle  $ACD$  are known, consequently the Ratio between  $AC$  and  $CD$  is known, for assuming  $CD$  to be Unity, or 1, then in the Triangle  $ACD$  by Trigonometry,

As the Sine of the Angle CAD	-	$56^\circ : 00'$	-	9.918574
Is to the Log. of the Side CD	-	1.	-	0.000000
So is the Radius	-	$90^\circ : 00'$	-	10.000000
				10.000000
				9.918574
To the Log. of the Side AC	-	1.21	-	0.081426

Hence we know the Sides  $AC$  and  $CD$  are as 1.21 to 1.



Calling  $m = 1.21$   $d = 1$ , therefore,

$$\left| \begin{array}{l} 1 \\ \hline \end{array} \right| m : d :: a : \frac{da}{m} = CD, \text{ as at the first} \\ \text{or second Steps, Question 95.}$$

Now BA being considered as the Base of the Triangle BAC, and CD as its Perpendicular, hence by the Rule for finding the Area of the Triangle,  $BA \times DC = 2b$ , that is in

Symbols	2	$2b = \frac{da}{m} \times BA.$
$2 \div \frac{da}{m}$	3	$\frac{2bm}{da} = BA, \text{ for } 2b \text{ or } \frac{2b}{1} \div \frac{da}{m}$ $= \frac{2bm}{da}$ by the Rule for Division of Vulgar Fractions.
And	4	$AD = \sqrt{aa - \frac{ddaa}{mm}}$ for the Tri- angle ACD is right-angled, where $a = AC$ ; and $\frac{da}{m} = CD.$
3 + 4	5	$\frac{2bm}{da} + \sqrt{aa - \frac{ddaa}{mm}} = BA +$ <span style="float: right;">(AD = BD)</span>
5 ⊖ 2	6	$\frac{4bbmm}{ddaa} + \frac{4bm}{da} \sqrt{aa - \frac{ddaa}{mm}}$ $+ aa - \frac{ddaa}{mm} = BD^2, \text{ or } BD$ <span style="float: right;">(squared)</span>
1 ⊖ 2	7	$\frac{ddaa}{mm} = CD^2, \text{ or } CD \text{ squared.}$
6 + 7	8	$\frac{4bbmm}{ddaa} + \frac{4bm}{da} \sqrt{aa - \frac{ddaa}{mm}}$ $+ aa = CB^2, \text{ or } CB \text{ squared.}$

Having now got an Expression equal to the Square of CB, we must endeavour to find another Expression for CB from some other Data.

Now the Sum of all the Sides is given, that is,

	9	$BC + AC + AB = s$
But	10	$AC = a$ , and $AB = \frac{2bm}{da}$ by the third Step.
9 . 10	11	$BC + a + \frac{2bm}{da} = s$
11 — $\frac{2bm}{da}$	12	$s - \frac{2bm}{da} = BC + a$
12 — $a$	13	$s - \frac{2bm}{da} - a = BC$
13 ⊙ 2	14	$ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} + \frac{4bbmm}{ddaa}$ $+ aa = \overline{BC}^2$ , or $BC$ squared.
8 . 14	15	$ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} + \frac{4bbmm}{ddaa}$ $+ aa = \frac{4bbmm}{ddaa} + \frac{4bm}{da} \sqrt{aa - \frac{ddaa}{mm}}$ $+ aa$
15 — $\frac{4bbmm}{ddaa}$	16	$ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} + aa$ $= \frac{4bm}{da} \sqrt{aa - \frac{ddaa}{mm}} + aa$
16 — $aa$	17	$ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} =$ $\frac{4bm}{da} \sqrt{aa - \frac{ddaa}{mm}}$

Here the Learner may observe that the unknown Quantity is under the radical Sign, and therefore as such Equations are generally squared to take away the Surd, the same is to be done here; but as it is  $aa$  in all the Quantities under the radical Sign, we can extract the square Root of  $aa$ , and join it to the rational Quantity, leaving the remaining Part of the *Surd* under the radical Sign, thus  $\frac{4bm}{da} \sqrt{aa - \frac{ddaa}{mm}} = \frac{4bm}{da} a \sqrt{1 - \frac{dd}{mm}}$   
 $= \frac{4bm}{d} \sqrt{1 - \frac{dd}{mm}}$ , whence the seventeenth Equation becomes

$$\begin{array}{l}
 18 \quad ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} = \\
 \quad \frac{4bm}{d} \sqrt{1 - \frac{dd}{mm}}, \text{ by which Means} \\
 \quad \text{we have saved the Trouble of squar-} \\
 \quad \text{ing the seventeenth Step.} \\
 18 \times da \quad 19 \quad ssda - 4sbm - 2dsaa + 4bma = \\
 \quad 4bma \sqrt{1 - \frac{dd}{mm}}
 \end{array}$$

The Equation being now cleared of its Fractions, it appears *quadratic*, for the Powers of  $a$  are only to the first and second Power, and the Surd is Part of one of the Co-efficients of  $a$ .

$$\begin{array}{ll} \cdot 19 \pm & 20 \\ 2ds & 21 \end{array}$$

$$2dsaa + 4bma\sqrt{1 - \frac{dd}{mm}} : -4bma - ssda$$

$$( = -4sbm )$$

$$4bma\sqrt{1 - \frac{dd}{mm}} : -4bma - ssda$$

$$aa + \frac{2ds}{2ds}$$

$$= -\frac{4sbm}{2ds} = -\frac{2bm}{d}$$

$$4bm\sqrt{1 - \frac{dd}{mm}} : -4bm - ssd$$

Now being a known Quantity, and = -45, put x = -45.

$$then \quad aa - xa = -\frac{2bm}{d}$$

$$22c \square \quad 23 \quad aa - xa + \frac{xx}{4} = \frac{xx}{4} - \frac{2bm}{d}$$

$$23w 2 \quad 24 \quad a - \frac{x}{2} = \sqrt{\frac{xx}{4} - \frac{2bm}{d}}$$

$$24 + \frac{x}{2} \quad 25 \quad a = \frac{x}{2} \pm \sqrt{\frac{xx}{4} - \frac{2bm}{d}} \text{ the Equation being ambiguous by the 22d Step.}$$

Whence we shall find  $a = 27.21 = AC$ .

And by the third Equation  $\frac{2b}{da} m = 17.78 = B A.$

And by the thirteenth Equation  $s - \frac{2bm}{da} - a = 40.01 = B C.$

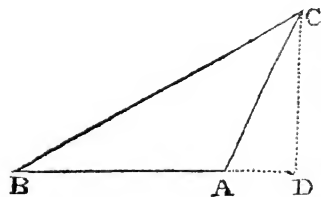
This being the most difficult Solution we have yet had, a Review or Summary Account of the Operation may not be useless to the Learner, in giving him some Idea how to begin and form a Judgment in such Cases.

Now because the Angles of the Triangle  $ACD$  are known, we have the *Ratio* of the Sides given, whence assuming  $CD$  as known, I find a proportional Number for  $AC$ , and from thence I can express  $CD$  and  $AC$  in Symbols, and  $CD$  being considered as the Perpendicular to the Triangle  $BAC$ , of which the Base is  $BA$ , from the Rule for finding the Area of a Triangle, I obtain an Expression for  $BA$ ; and I express  $AD$  in Symbols, from knowing  $AC$  and  $CD$ , and adding  $AD$  to  $BA$ , I have Expressions for  $BD$  and  $DC$ , each of which being squared, their Sum is equal to the Square of  $BC$ .

Then from some other *Data* I find an Expression for  $BC$ , and because the Sum of the Sides is given, and having Expressions for the two Sides  $BA$  and  $AC$ , it is easy to find an Expression for  $BC$ , as at the thirteenth Step, which being squared, is made equal to the former Square of  $BC$ , and the Equation is reduced, as in the Work.

This and several other Questions are taken from Sir ISAAC NEWTON, the perpetual and everlasting Honour, Ornament, and Glory of our Nation; and I have only endeavoured to accommodate his Solutions to the Learner, in explaining them in a more copious Manner.

Question 98. *In the Triangle  $ABC$ , there is given the Altitude  $CD = 7$ , the Base  $AB = 10$ , and the Sum of the Sides  $BC + AC = 23$ . To find the Sides of the Triangle?*



Because the three Sides of the Triangle  $BAC$  will be easily expressed in Symbols, and the Triangle  $BCD$  being right-angled, we shall easily find to what  $BD$  is equal.

Again, as the Triangle  $ACD$  is right-angled, and the Sides  $AC$  and  $CD$  are known in Symbols, therefore  $AD$  is known in Symbols.

Now if from  $BD$  before found in Symbols, we subtract  $BA$ , there remains another Value for  $AD$ , which being made equal to the former, we have an Equation, which is sufficient, if we use but one unknown Quantity.

And

And as here will be a new Method of expressing the Quantities sought, I refer the Reader to Question 41, where at Step 8. he will find, that in any two Numbers, if their Difference is subtracted from their Sum, and the Remainder divided by 2, the Quotient will be equal to the lesser; and at the same Question, if he exterminates  $e$  and finds  $a$ , or the greater Number, it will be equal to the Sum and Difference of the two Numbers added together and divided by 2.

Therefore put  $x = CD = 7$ ,  $b = BA = 10$ ,  $c =$  half the Sum of the Sides  $BC + AC = 11.5$  and  $a =$  half their Difference, then the greater Side or  $BC = c + a$ , and the lesser Side or  $AC = c - a$ , now in the Triangle  $BCD$

by 47 . e . 1	1	$\sqrt{cc + 2ca + aa - xx} = BD$ , for $BC = c + a$ , and $CD = x$ And in the Triangle $ACD$ ,
by 47 . e . 1	2	$\sqrt{cc - 2ca + aa - xx} = AD$ , for $AC = c - a$ , and $CD = x$
but	3	$BA = b$
1 - 3	4	$\sqrt{cc + 2ca + aa - xx} - b = BD$ $- BA = AD$
2 . 4	5	$\sqrt{cc - 2ca + aa - xx} = \sqrt{cc + 2ca + aa - xx} - b$
5 $\oplus$ 2	6	$cc - 2ca + aa - xx = cc + 2ca + aa - xx - 2b\sqrt{cc + 2ca + aa - xx} + bb$ By transposing the Quantities which destroy one another
we have	7	$-4ca = -2b\sqrt{cc + 2ca + aa - xx} + bb$
7 $\pm$	8	$2b\sqrt{cc + 2ca + aa - xx} = bb + 4ca$
8 $\oplus$ 2	9	$4bbcc + 8bbca + 4bbaa - 4bbxx = bbb + 8bbca + 16ccaa$
9 - 8bbca	10	$4bbcc + 4bbaa - 4bbxx = bbb + 16ccaa$
10 - bbbb	11	$16ccaa = 4bbcc + 4bbaa - 4bbxx - bbb$
11 - 4bbaa	12	$16ccaa - 4bbaa = 4bbcc - 4bbxx - bbb$
12 $\div$ 16cc - 4bb	13	$aa = \frac{4bbcc - bbb - 4bbxx}{16cc - 4bb} = \frac{bb}{4}$ $- \frac{bbxx}{4cc - bb}$ . See Page 276.
13 $\text{w}$ 2	14	$a = \sqrt{\frac{bb}{4} - \frac{bbxx}{4cc - bb}} =$ $b\sqrt{\frac{1}{4} - \frac{xx}{4cc - bb}} = 3.69$

Whence  $c + a = 15.19 = BC$ , and  $c - a = 7.81 = AC$ .

If the Learner should be perplexed to see the Contractions at the thirteenth and fourteenth Steps, they may be illustrated thus

$$\frac{4bbcc - bbbb - 4bbxx}{16cc - 4bb} = \frac{4bbcc - bbbb}{16cc - 4bb} - \frac{4bbxx}{16cc - 4bb}$$

for it is the same Thing whether the Quantities that compose the Numerator, are placed successively one after another like one continued Fraction, or placed separately and distinctly, like different Fractions, the Quantities that compose the Denominator being placed under each distinct Numerator.

$$\begin{array}{r} \text{But } 16cc - 4bb) 4bbcc - bbbb \left( \frac{bb}{4} \right. \\ \underline{4bbcc - bbbb} \\ 0 \end{array}$$

The Quotient Quantity is  $bb$ , and as the Co-efficients of the Divisor are respectively four Times more than those of the Dividend, therefore under the Quotient Quantity  $bb$  place 4, and  $\frac{bb}{4}$  is the Quotient exact.

And this Fraction  $\frac{4bbxx}{16cc - 4bb} = \frac{bbxx}{4cc - bb}$ , for it is only dividing the Co-efficients by 4, therefore the Contractions are as at the thirteenth Step.

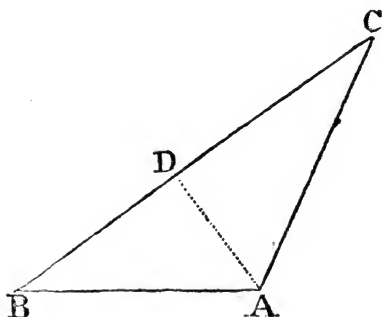
The Contractions at the fourteenth Step, arise from its being  $bb$  in all the Terms under the radical Sign, for it is only placing  $b$  the square Root of  $bb$  without the radical Sign, by which means  $\sqrt{\frac{bb}{4}}$ , or  $\sqrt{\frac{1bb}{4}}$  is  $b\sqrt{\frac{1}{4}}$ .

Question 99. In the Triangle  $BCA$ , there is given the Base  $AB = 6$ , the Sum of the Sides  $AC + BC = 18$ , and the vertical Angle at  $C = 30^\circ : 00'$ . To find the Sides  $AC$  and  $BC$ .

Let fall the Perpendicular  $AD$ , and in the Triangle  $ACD$ , because the Angle at  $C$  is given, all the Angles of that Triangle are known, and therefore the Ratio of the Sides is known, by which means we can get an Expression for  $CD$ .

And

And because  $AD$  is a Perpendicular that falls within the Triangle, and the Angle at  $C$  is acute, therefore by 13. *e. 2*,  $BC$  squared added to  $AC$  squared, is equal to  $BA$  squared added to the Product of  $2 BC \times CD$ , from whence we shall have another Expression for  $CD$ . Now if we can express the Sides of the Triangle with one unknown Quantity, this Equation between the two Values of  $CD$  will be sufficient.



In the Triangle  $ACD$ , because the Angle at  $C$  is known, and  $AD$  being a Perpendicular to  $CB$ , all the Angles of the Triangle  $ACD$  are known, therefore assuming  $CD = 1$ , by Trigonometry,

As the Sine of the Angle $CAD$	- $60^\circ : 00'$	-	9.937531
Is to the Log. of the Side $CD$	- 1.	-	0.000000
So is the Sine of the Angle $CDA$	- $90 : 00$	-	10.000000
			10.000000
			9.937531

To the Log. of the Side $AC$	- 1.15	-	0.062469
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Hence we know that as 1.15 is to 1, so is  $AC$  to  $CD$ .

Then let  $AB = 6 = x$ , half the Sum of the Sides  $AC + BC = 9 = b$ , and half their Difference  $= a$ , then as in the last Question, the greater Side or  $BC = b + a$ , and the lesser Side or  $AC = b - a$ ,  $d = 1.15^n = 1$ .

Because  $AC$  is to  $CD$ , as 1.15 is to 1.

Therefore	1	$d : n :: b - a : \frac{nb - na}{d} = CD$
by 13. e. 2	2	$bb + \frac{2ba + aa + bb - 2ba + aa}{2b + 2a} = xx$
$2 - xx$	3	$2bb + 2aa - xx = 2b + 2a \times CD$
$3 \div 2b + 2a$	4	$\frac{2bb + 2aa - xx}{2b + 2a} = CD$

The short Line over the two Quantities  $2b + 2a$  in the second and third Equations, signifies they are both to the multiplied into CD, otherwise there would be no Distinction whether CD is to be multiplied into  $2a$  only, or all the Quantities on that Side of the Equation.

Now make an Equation between the two Values of CD found at the first and fourth Equations.

1. 4	5	$\frac{2bb + 2aa - xx}{2b + 2a} = \frac{nb - na}{d}$
$5 \times 2b + 2a$	6	$2bb + 2aa - xx = \frac{2nbb - 2bna + 2bna - 2naa}{d}$
$6 \times d$	7	$2d bb + 2d aa - dxx = 2nbb - 2naa$
$7 + 2naa$	8	$2naa + 2d bb + 2d aa - dxx = 2nbb$
$8 + dxx$	9	$2naa + 2d bb + 2d aa = 2nbb + dxx$
$9 - 2d bb$	10	$2naa + 2d aa = 2nbb + dxx - 2d bb$
$10 \div 2n + 2d$	11	$aa = \frac{2nbb + dxx - 2d bb}{2n + 2d}$
$11 \text{ w } 2$	12	$a = \sqrt{\frac{2nbb + dxx - 2d bb}{2n + 2d}} = 1.99$

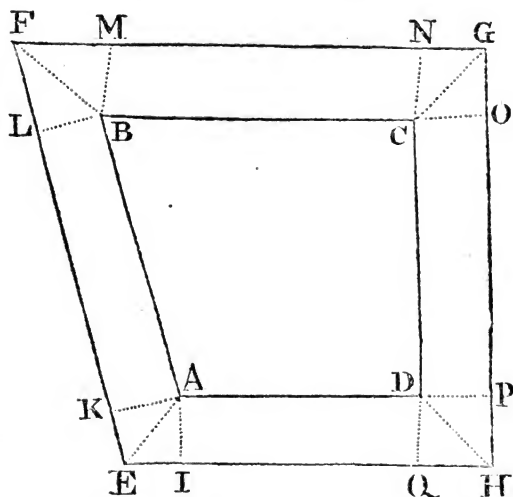
Whence  $BC = b + a = 10.99$  and  $AC = b - a = 7.01$

Question 100. In the Fish Pond ABCD, there is given the Side  $AD = 3$ ,  $DC = 35$ ,  $CB = 40$ , and  $AB = 38$ ; the Angle at  $A = 113^\circ$ , the Angle at  $B = 60^\circ$ , the Angle at  $C = 100^\circ$ , and the Angle at  $D = 87^\circ$ , and the Fish Pond is to be surrounded with an Area of 700, and every where of the same Breadth. To find the Breadth of the Walk?

Supposing the Walk to be drawn round the Pond as in the Figure, let fall the Perpendiculars AK, BL, BM, CN, CO, DP, DQ and AI, by which the Walk is divided into four Parallelograms AKLB, BMNC, COPD, DQIA, and



and into four Trapezia AIEK, BLFM, CNGO and DPHQ, and the Area of these four Parallelograms and four Trapezia is equal to the given Area of the Walk.



Let the Breadth of the Walk be  $a$ , and the Sum of the Sides  $AD + DC + CB + BA = 143 = b$ , then the Area of the four Parallelograms will be  $= ba$ . Let  $x = 700$ .

Draw  $AE$ ,  $BF$ ,  $CG$  and  $DH$ , because the Triangles  $AIE$  and  $AKE$  are equal, therefore the Angle  $AEK$  and  $AEI$  are equal, and each of these Angles are equal to half the Angle at  $A$  which is  $113^\circ$ , hence the Angle  $AEI$  is  $56^\circ : 30'$ .

Then in the Triangle  $AEI$  all the Angles are known, and consequently from plain Trigonometry, we can find the Ratio of the Sides  $EI$  and  $IA$ , for assuming  $EI$  to be *Unity*, or  $1$ , we have

As the Sine of the Angle $EAI$	-	$33^\circ : 30'$	-	9.741889
Is to the Log. of the Side $EI$	-	1.0	-	0.000000
So is the Sine of the Angle $AEI$	-	$56^\circ : 30'$	-	9.921107
				9.921107
				9.741889

To the Log. of the Side  $AI$  - - 1.51 - - 0.179218  
Hence

Hence we know that AI is to EI as 1.51 is to 1.

Then let  $d = 1.51$  and  $e = 1$ .

Hence  $d : e :: a : \frac{ea}{d} = EI$ , which being the Base of the Triangle EIA,

$$\begin{array}{l|l} \text{Hence} & 2 \left| \frac{ea}{d} \times \frac{a}{2} = \frac{eaa}{2d} = \text{the Area of the} \right. \\ & 2 \times \frac{1}{2} \left| 3 \left| \frac{eaa}{d} = \text{the Area of the Trapezium} \right. \right. \\ & & & \text{(Triangle EIA, and} \\ & & & \text{(EIAK.} \end{array}$$

Now in the Trapezium BLFM, because the Angle B is  $60^\circ$ , we have the Angle LFB =  $30^\circ$ , for the same Reasons as before; whence in the Triangle LFB, if we assume BL to be *Unity*, or 1, we shall find the proportional Number for FL to be 1.73 hence as 1 is to 1.73 so is BL to LF, let  $f = 1.73$

$$\begin{array}{l|l} \text{Then} & 4 \left| e : f :: a : \frac{af}{e} = LF \right. \\ & 4 \times \frac{a}{2} \left| 5 \left| \frac{af}{e} \times \frac{a}{2} = \frac{aaf}{2e} = \text{the Area of the} \right. \right. \\ & & & \text{(Triangle BLF.} \\ & 5 \times \frac{1}{2} \left| 6 \left| \frac{aaf}{e} = \text{the Area of the Trapezium} \right. \right. \\ & & & \text{(BLFM.} \end{array}$$

Again, in the Trapezium CNGO, because the Angle at C is  $100^\circ$ , the Angle CGN is  $50^\circ$ , for the same Reason as before; and assuming *Unity* for NG, we shall find 1.19 to be the proportional Number for NC, whence we know that CN and NG are as 1.19 is to 1, let  $g = 1.19$

$$\begin{array}{l|l} \text{Then} & 7 \left| g : e :: a : \frac{ea}{g} = NG, \text{ which being} \right. \\ & & & \text{the Base of the Triangle CNG,} \\ \text{Hence} & 8 \left| \frac{ea}{g} \times \frac{a}{2} = \frac{eaa}{2g} = \text{the Area of the} \right. \\ & & & \text{(Triangle CNG.} \\ & 8 \times \frac{1}{2} \left| 9 \left| \frac{eaa}{g} = \text{the Area of the Trapezium} \right. \right. \\ & & & \text{(CNGO.} \end{array}$$

Lastly, in the Trapezium DPHQ, because the Angle D is  $87^\circ$ , the Angle DHP is  $43^\circ : 30'$ , for the same Reason as before;

before; and assuming *Unity* for DP we shall find 1.07 to be the proportional Number for PH, hence we know that as 1 is to 1.07 so is DP to PH, let  $s = 1.07$

Then	10	$e : s :: a : \frac{as}{e} = PA$ , then as before
$10 \times \frac{a}{2}$	11	$\frac{as}{e} \times \frac{a}{2} = \frac{aas}{2e}$ the Area of the Triangle (angle DPH, hence
$11 \times 2$	12	$\frac{aas}{e} =$ the Area of the Trapezium (DPHQ.
	13	But it was before found that $ba =$ the Area of the four Parallelograms.

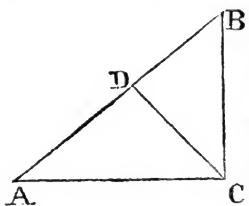
Now collect the Area of the Trapezia and Parallelograms into one Sum, and make them equal to the given Area of the Walk.

$3+6+9+12+13$	14	$\frac{eaa}{d} + \frac{aaf}{e} + \frac{eaa}{g} + \frac{aas}{e} + ba$ $= x = 700$
substitute	15	$p = \frac{e}{d} + \frac{f}{e} + \frac{e}{g} + \frac{s}{e}$ , the Coefficients of $aa = 4.302$
$14 \cdot 15$	16	$paa + ba = x$
$16 \div p$	17	$aa + \frac{ba}{p} = \frac{x}{p}$
$17 \cdot \square$	18	$aa + \frac{ba}{p} + \frac{bb}{4pp} = \frac{bb}{4pp} + \frac{x}{p}$
$18 \text{ w } 2$	19	$a + \frac{b}{2p} = \sqrt{\frac{bb}{4pp} + \frac{x}{p}}$
$19 - \frac{b}{2p}$	20	$a = \sqrt{\frac{bb}{4pp} + \frac{x}{p}} - \frac{b}{2p} = 4.35$ the Breadth of the Walk.

Because the Angles and Sides of the Fifth Pond are given, the Figure may be drawn; but for the Ease of the Numerical Calculation I have chose such Numbers, as will not exactly agree with a Geometrical Figure.

Question 101. In the right-angled Triangle ABC, given the Perimeter or Sum of the Sides  $AC + CB + AB = 24$ , and the Perpendicular

Perpendicular  $CD = 4.75$  let fall from the Right-angle at  $C$  upon the Hypothenufe  $AB$ . To find the Sides of the Triangle?



Let  $CD = b = 4.75$   $AB + BC + CA = x = 24$ ,  $AB = a$ , then the Sum of two of the Sides, or  $AC + CB = x - a$ , and as at Question 98 let  $y =$  the Difference of the same two Sides  $AC$  and  $CB$ . And because the greater Number or Leg is equal to the Sum and Difference of the two Numbers or Legs divided by 2, as in the last Question,

therefore  $AC$  the greater Leg  $= \frac{x - a + y}{2}$ , and  $BC$  the lesser

$$\text{Leg} = \frac{x - a - y}{2}.$$

Having Expressions for  $AB$ ,  $BC$ , and  $AC$ , the three Sides of the Triangle  $ABC$ , in which there are two unknown Quantities  $a$  and  $y$ , we must raise two Equations from the Properties of the Figure, and because the Triangles  $ABC$ ,  $BCD$  are similar, therefore by

4 . e . 6	1	$AB : BC :: AC : CD$
In Symbols	2	$a : \frac{x - a - y}{2} :: \frac{x - a + y}{2} : b$
whence	3	$ab = \frac{xx - xa + xy - xa + aa - ay - xy + ay - yy}{4}$
that is	4	$ab = \frac{xx - 2xa + aa - yy}{4}$
		And because the Triangle $ACB$ is right-angled,
by 47 . e . 1	5	$\frac{xx - 2xa + 2xy + 2ay + aa + yy}{4}$
		$+ \frac{xx - 2xa + 2xy - 2ay + aa + yy}{4} = aa$
5 contracted.	6	$\frac{2xx - 4xa + 2aa + 2yy}{4} = aa$

Hence the two Equations which contain the Question are the fourth and sixth, and as  $y$  is only to the Square in each of them, find in both the Value of  $yy$ . But the sixth Equation becomes

$$7 \mid xx$$

	7	$\frac{xx - 2xa + aa + yy}{2} = aa$
$7 \times 2$	8	$xx - 2xa + aa + yy = 2aa$
$8 \pm$	9	$yy = aa + 2xa - xx$
$4 \times 4$	10	$4ab = xx - 2xa + aa - yy$
$10 \pm 11$	11	$yy = xx - 2xa + aa - 4ab$
$9 \cdot 11$	12	$aa + 2xa - xx = xx - 2xa + aa - 4ab$
$12 - aa$	13	$2xa - xx = xx - 2xa - 4ab$
$13 + 4ab$	14	$2xa + 4ab - xx = xx - 2xa$
$14 + 2xa$	15	$4xa + 4ab - xx = xx$
$15 + xx$	16	$4xa + 4ab = 2xx$
$16 \div 4x + 4b$	17	$a = \frac{2xx}{4x+4b} = \frac{xx}{2x+2b} = 10 = AB.$
		Now $a$ being found, therefore
$9 \text{ w } 2$	18	$y = \sqrt{aa + 2xa - xx} = 2.$
thence	19	$AC = \frac{x-a+y}{2} = 8.$
and	20	$BC = \frac{x-a-y}{2} = 6.$

Question 102. In the right-angled Triangle ABC, given the Hypotenuse AB = 10, and the Sum of the Sides and Perpendicular CD, that is AC + CB + CD = 18.75 To find the Sides AC and BC? Vide last Figure.

Let AB =  $b = 10$ ,  $x = 18.75$  CD =  $a$ , then AC + CB =  $x - a$ ; now put  $y =$  the Difference between the Legs AC and CB, then as in the two last Questions AC, or the greater Leg is  $\frac{x-a+y}{2}$ , and the lesser Leg, or CB =  $\frac{x-a-y}{2}$ .

Having expressed the Sides of the Triangle ABC in Symbols, in which there are two unknown Quantities  $a$  and  $y$ , we must raise two Equations from the Properties of the Figure, and because ABC is a right-angled Triangle, therefore

$$\text{by 47. e. 1} \quad \left| \begin{array}{l} \frac{xx - 2xa + 2xy - 2ay + aa + yy}{4} \\ + \frac{xx - 2xa - 2xy + 2ay + aa + yy}{4} = bb \end{array} \right|$$

The two Triangles ABC and CBD being similar, therefore

by 4. e. 6	2	AB:AC::CB:CD, that is in Symbols,
	3	$b : \frac{x-a+y}{2} :: \frac{x-a-y}{2} : a$
3 ∴	4	$ba = \frac{xx-xa+xy-xa+aa-ay-xy+ay-yy}{4}$
		Hence the Question is contained in the first and fourth Equations.
1 contracted	5	$\frac{xx-2xa+aa+yy}{2} = bb$
4 contracted	6	$ba = \frac{xx-2xa+aa-yy}{4}$

Now in both these Equations find the Value of  $yy$ , there being no other Power of  $y$ .

$5 \times 2$	7	$xx-2xa+aa+yy=2bb$
$7 \pm$	8	$yy=2bb-aa+2xa-xx$
$6 \times 4$	9	$4ba=xx-2xa+aa-yy$
$9 \pm$	10	$yy=xx-2xa+aa-4ba$
$8 \cdot 10$	11	$xx-2xa+aa-4ba=2bb-aa+2xa-xx$
$11 + aa$	12	$xx-2xa+2aa-4ba=2bb+2xa-xx$
$12 - xx$	13	$-2xa+2aa-4ba=2bb+2xa-2xx$
$13 - 2xa$	14	$2aa-4xa-4ba=2bb-2xx$
$14 \div 2$	15	$aa-2xa-2ba=bb-xx$
		Because $xx$ is greater than $bb$ , therefore the Equation is <i>ambiguous</i> .
substitute then	16	$-z = -2x - 2b = -57.5$
	17	$aa - za = bb - xx$
$17 \square$	18	$aa - za + \frac{zz}{4} = bb - xx + \frac{zz}{4}$
$18 \text{ as } 2$	19	$a - \frac{z}{2} = \sqrt{bb - xx + \frac{zz}{4}}$
$19 + \frac{z}{2}$	20	$a = \frac{z}{2} \pm \sqrt{bb - xx + \frac{zz}{4}} = 4.78$ (or 52.72)
by the 10th Step	21	$y = \sqrt{xx - 2xa + aa - 4ba} = 1.99$

Then

Then  $AC = \frac{x-a+y}{2} = 7.98$  and  $CB = \frac{x-a-y}{2} = 6$ .

In the above Equation where  $a = 4.78$  or  $52.72$  the Value of  $a$  must be  $4.78$  for it cannot be  $52.72$  as the Sum of the three Quantities is only  $18.75$ .

Question 103. In the right-angled Triangle ABC, there is given the Sum of the Sides  $AC + BC = 14$ , and the Perpendicular  $CD = 4.75$  To find the Sides of the Triangle? See Figure, Question 101.

Let  $AC + BC = x = 14$ ,  $AC - BC = y$  the Difference of the Sides, then, as in the preceding Questions, the greater Side or  $AC = \frac{x+y}{2}$ , and the lesser Side or  $BC = \frac{x-y}{2}$ .

Put  $AB = a$  and  $DC = b = 4.75$

Having expressed all the Sides of the Triangle ABC in Symbols, amongst which two are unknown, viz.  $a$  and  $y$ , we must raise two Equations from the Figure, then because the Triangle ABC is right-angled, therefore

by 47 . e . 1	1	$\frac{xx + 2xy + yy}{4} + \frac{xx - 2xy + yy}{4} = aa$
that is	2	$\frac{xx + yy}{2} = aa$
		Because the Triangles ABC and CBD are similar,
by 4 . e . 6	3	$AB : AC :: CB : CD$
in Symbols	4	$a : \frac{x+y}{2} :: \frac{x-y}{2} : b$
4 . .	5	$ba = \frac{xx - yy}{4}$

Hence the Question is contained in the second and fifth Equations, and because there are no other Powers of  $y$  but  $yy$  in either of those two Equations, find the Value of  $yy$  in both Equations.

$2 \times 2$	6	$xx + yy = 2aa$
$6 - xx$	7	$yy = 2aa - xx$
$5 \times 4$	8	$4ba = xx - yy$
$8 + yy$	9	$yy + 4ba = xx$

9 - 4 b a	10	$yy = xx - 4ba$
7 . 10	11	$2aa - xx = xx - 4ba$
11 $\pm$	12	$2aa + 4ba = 2xx$
12 $\div 2$	13	$aa + 2ba = xx$
13 c $\square$	14	$aa + 2ba + bb = xx + bb$
14 w 2	15	$a + b = \sqrt{xx + bb}$
15 - b	16	$a = \sqrt{xx + bb} - b = 10.03$ or neglecting the Fraction $AB = 10.$
7 w 2	17	$y = \sqrt{2aa - 2x} = 2$

Then  $AC = \frac{x+y}{2} = 8$ , and  $BC = \frac{x-y}{2} = 6$ .

*The same Question done in another Manner.*

Let  $AC + BC = x = 14$ ,  $AC = a$ , then  $BC = x - a$ ,  
 $CD = b = 4.75$  and because the Triangle  $ABC$  is right-  
 angled, therefore  $AB = \sqrt{xx - 2xa + 2aa}$ .

Here we have Expressions for all the Sides of the Triangle,  
 with only one unknown Quantity, and therefore one Equation  
 will be sufficient. And as the Triangles  $ABC$  and  $CBD$  are  
 similar, therefore

by 4 . e . 6	1	$AB : AC :: CB : CD$
in Symbols	2	$\sqrt{xx - 2xa + 2aa} : a :: x - a : b$
2 . .	3	$b\sqrt{xx - 2xa + 2aa} = xa - aa$

Square both Sides of the Equation, the unknown Quantity  
 being under the radical Sign.

3 $\odot$ 2	4	$bbxx - 2bbxa + 2bbba = xxa$ $- 2xaaa + aaaa$
		Ranging the Equation according to the Powers of the unknown Quantity.
4 $\pm$	5	$aaaa - 2xaaa + xxa - 2bbba$ $+ 2bbxa = bbxx$

Tho' the Equation here appears as if *adfect*ed, yet it may be  
 resolv'd by *completing the square*, as in Quadratics.

And to give the Learner a clear Idea how this is done, if he  
 squares any three Quantities  $m - n - z$ , in the Square he  
 will find six Terms,  $mm + nn + zz - 2mn - 2mz + 2nz$ ,  
three



three being pure Powers of the Quantities squared, and the other three will be double Rectangles, or Products of these Quantities, and therefore any Expression that comes under these Circumstances, may have its square Root extracted.

Now  $aaaa$  is the Square of  $a$   
 And  $xxaa$  is the Square of  $xa$

And  $2xaaa$  is the double Rectangle, or Product of these Roots.

And  $2bbaa$  is the double Rectangle, or Product of  $bb \times aa$ .

And  $2bbxa$  is the double Rectangle, or Product of  $bb \times xa$ .

From hence it appears that the above Equation of five Quantities has two of them,  $aaaa$  and  $xxaa$ , whose square Roots may be taken, and that the other three Quantities are double Rectangles of those two Roots,  $aa$  and  $xa$ , and a third Quantity  $bb$ , therefore multiply this Quantity  $bb$  by itself, and add the Product  $bbbb$  to both Sides of the Equation, which makes it a compleat Square, thus,

	6	$aaaa - 2xaaa + xxaa - 2bbaa + 2bbxa$ $+ bbbb = bbbb + bbbx$
6 w 2	7	$aa - xa - bb = \sqrt{bbbb + bbbx}$ $= b\sqrt{bb + xx}$
7 + bb	8	$aa - xa = bb + b\sqrt{bb + xx}$
8 c □	9	$aa - xa + \frac{xx}{4} = \frac{xx}{4} + bb + b\sqrt{bb + xx}$
9 w 2	10	$a - \frac{x}{2} = \sqrt{\frac{xx}{4} + bb + b\sqrt{bb + xx}}$
10 + $\frac{x}{2}$	11	$a = \frac{x}{2} + \sqrt{\frac{xx}{4} + bb + b\sqrt{bb + xx}}$ $= 18.9 = A C$ , a different Value of what it had before, for then it was only 8.

To explain this to the Learner, if he extracts the Square Root of  $aa - 2xa + xx$ , he will find it to be  $a - x$ , or  $x - a$ , the double Rectangle, viz.  $2ax$  having the Sign  $-$  we are sure either  $a$ , or  $x$  must be negative; but in this Case we are to determine which is to be negative by the Consequences that follow, for if there follows an Impossibility in supposing  $a - x$  to be the Root, then the Root must be  $x - a$ .

To apply this to the Square before us at the sixth Step, viz.  $aaaa - 2xaaa + xxaa - 2bbaa + 2bbxa + bbbb$ .

Now the square Root of  $aaaa$  is  $-$   $-$   $aa$   
 And the square Root of  $xxaa$  is  $-$   $-$   $xa$   
 And the square Root of  $bbbb$  is  $-$   $-$   $bb$

But as  $2xaaa$  the double Rectangle, or Product of  $aa \times xa$  has the Sign  $-$ , therefore it must be in the square Root either  $aa - xa$ , or  $xa - aa$ ; but as an Impossibility attends putting it  $aa - xa$ , we now put it  $xa - aa$ , and to determine what Sign  $bb$  must have in the Root, now the double Product  $2bbaa$  having the Sign  $-$ , therefore it must be  $+bb$  or  $bb$ , as  $2bbx - aa$  produces  $-2bbaa$ , then taking the

sixth Equation	7	$aaaa - 2xaaa + xxaa - 2bbaa$
		$+ 2bbxa + bbbb = bbbb + bbxx$
7 $uu$ 2	8	$xa - aa + bb = \sqrt{bbbb + bbxx}$
		$= b\sqrt{bb + xx}$
		Because $aa$ is negative transpose it,
8 $+aa$	9	$aa + b\sqrt{bb + xx} = xa + bb$
9 $-xa$	10	$aa - xa + b\sqrt{bb + xx} = bb$
10 $-b\sqrt{bb + xx}$	11	$aa - xa = bb - b\sqrt{bb + xx}$

Here the Equation appears *quadratic*, and because  $-b\sqrt{bb + xx}$  is greater than  $bb$ , it is likewise *ambiguous*.

11 $c \square$	12	$aa - xa + \frac{xx}{4} = \frac{xx}{4} + bb - b\sqrt{bb + xx}$
12 $uu$ 2	13	$a - \frac{x}{2} = \sqrt{\frac{xx}{4} + bb - b\sqrt{bb + xx}}$
13 $+$ $\frac{x}{2}$	14	$a = \frac{x}{2} \pm \sqrt{\frac{xx}{4} + bb - b\sqrt{bb + xx}}$
		$= 7 \pm 1.11 = 8.11 \text{ or } 5.89 = AC.$

Question 104. In the right-angled Triangle ABC, there is given the Sum of the Legs  $AC + BC = 14$ , and the Sum of the Hypotenuse and Perpendicular  $AB + CD = 14.75$  To find the Sides of the Triangle? See Figure, Question 101.

Let  $AC + BC = 14 = x$ ,  $AB + CD = 14.75 = b$ ,  $AC = a$ ,  $AB = y$ , then  $BC = x - a$ , and  $CD = b - y$ .

Having now expressed the Sides of the Triangle in Symbols, in which there are two that are unknown, therefore raise two Equations from the Properties of the Figure.

And

And because the Triangle ABC is right-angled, therefore by

$$47 \text{ e. 1} \quad | \quad 1 \quad | \quad xx - 2xa + 2aa = yy$$

And because the Triangles ABC and CBD are similar, therefore by

$$\begin{array}{l|l|l} 4 \text{ e. 6} & 2 & AB : AC :: BC : CD \\ \text{in Symbols} & 3 & y : a :: x - a : b - y \\ 3 \text{ . .} & 4 & by - yy = xa - aa \end{array}$$

Now both the unknown Quantities being to the first and second Power, in the fourth Equation, and it being  $yy$  only in the first Equation, and these two Equations containing the Conditions of the Question, find the Value of  $yy$  in each Equation.

$$\begin{array}{l|l|l} 4 \pm & 5 & yy = by + aa - xa \\ 1 \cdot 5 & 6 & by + aa - xa = xx - 2xa + 2aa \\ 6 - aa & 7 & by - xa = xx - 2xa + aa \\ 7 + xa & 8 & by = xx - xa + aa \\ 8 \div b & 9 & y = \frac{xx - xa + aa}{b} \end{array}$$

Raise this Equation to the second Power, and make it = to the first Equation, as there it is only  $yy$ ; whereas in the fourth Equation, if we were to exterminate  $y$ , we must use the Values of  $y$  and  $y y$ .

$$\begin{array}{l|l|l} 9 \odot 2 & 10 & yy = \frac{xxxx - 2axxx + 2aaxx - 2aaax + aaxx + aaaa}{bb} \\ 1 \cdot 10 & 11 & \frac{xxxx - 2axxx + 2aaxx - 2aaax + aaxx + aaaa}{bb} = \\ & & xx - 2xa + 2aa \\ 11 \times bb & 12 & xxxx - 2axxx + 2aaxx - 2aaax + aaxx + aaaa \\ & & = xxbb - 2bbxa + 2bbba \end{array}$$

Transposing and ranging the Equation, according to the highest Dimensions of the unknown Quantity.

$$12 \pm \quad | \quad 13 \quad | \quad aaaa - 2xaaa + 2aaxx - 2bbba + aaxx - 2axxx + 2bbxa + xxxx - xxbb = 0$$

Tho' the Equation now appears to be *affected*, yet the square Root may be compleated, as in the last.

To show the Learner how this is to be done, if he squares any four Quantities, (for the Root of the above Equation will consist

of so many Quantities) he will find ten Terms in the Square, four of which are pure Powers of the Quantities that were squared, and the other six will be double Rectangles of those Quantities, of which each particular Root will constitute a Part of three of the Rectangles.

Now in the above Equation  $aaaa, aaxx, xxxx,$

Are the Squares of  $a, x,$

And the Quantities  $2xaaa, 2aaxx, 2axxx,$  are the double Rectangles of those Parts, or Roots.

And by examining  $2bbba, 2bbxa, xxxb,$  the remaining Terms in the above Equation, the first two are double Rectangles of  $bb \times aa$  and  $bb \times ax$ , but the last Term is only a single Rectangle of  $bb \times xx$ , therefore to compleat the Square there wants  $-xxbb$ , which when added to  $-xxbb$ , will make that a double Rectangle of  $bb \times xx$ , and as we have no pure Power of  $bb$ , which being squared is  $bbbb$ , hence if we add  $-bbxx + bbbb$  to our Equation, we shall make it a Square, therefore

$$\left| \begin{array}{l} 14 \mid aaaa - 2xaaa + aaxx - 2bbba + aaxx \\ \quad - 2axxx + 2bbxa + xxxx - 2xxbb \\ \quad + bbbb = bbbb - xxbb \end{array} \right.$$

Before we proceed, perhaps the Learner might have observed that  $xxbb$  is the Square of  $xb$ , and therefore might suppose that to be one of the Roots, but then he will find  $xb$  to make a Part only of two of the Rectangles, whereas, if it had been one of the Roots, it would have made a Part of three of the Rectangles. Besides, if  $xb$  had been one of the Roots, it must have had the Sign  $+$  in the Square, by Art. 34.

$$14 \text{ m } 2 \mid 15 \mid \begin{array}{l} aa - ax + xx - bb = \sqrt{bbbb - xxbb} \\ \quad = b\sqrt{bb - xx} \end{array}$$

The Manner of extracting the Root is thus, I first extract the square Root of  $aaaa$ , which is  $aa$ , then the square Root of  $aaxx$ ; the next pure Power is  $ax$ , and to determine whether  $ax$  must have the Sign  $+$  or  $-$ , observe the Sign of the double Rectangle of these two Roots, viz. of  $2xaaa$ , which because it is  $-$ , I therefore in the Root make it  $-ax$ .

The next pure Power is  $xxxx$ , whose Root is  $xx$ , then observe the Sign of the double Rectangle of this and one of the two former Roots, as of  $2aaxx$ , which being  $+$ , and the Root  $aa$  being  $+$ , therefore in the Root make it  $+xx$ .

The

The last pure Power is  $bbbb$ , whose Root is  $bb$ , then observe the Sign of the double Rectangle of this and one of the former Roots, as the last Root  $xx$ , but the double Rectangle of these is  $2xxbb$ , which being *negative*, and the Sign of  $xx$  being  $+$ , therefore place the Sign  $-$  before  $bb$ .

15 + $bb$	16	$aa - ax + xx = bb + b\sqrt{bb - xx}$
16 - $xx$	17	$aa - ax = bb - xx + b\sqrt{bb - xx}$
17 c $\square$	18	$aa - ax + \frac{xx}{4} = bb - \frac{3xx}{4} +$ $b\sqrt{bb - xx}, \text{ for } \frac{xx}{4} - xx = -\frac{3xx}{4}$
18 w 2	19	$a - \frac{x}{2} = \sqrt{bb - \frac{3xx}{4}} + b\sqrt{bb - xx}$
19 + $\frac{x}{2}$	20	$a = \frac{x}{2} + \sqrt{bb - \frac{3xx}{4}} + b\sqrt{bb - xx}$

$= 18.79 = AC$ , which is impossible, for  $AC + BC = 14$  by the Question, consequently  $AC$  cannot be 18.79

This impossible Conclusion is owing to taking the Root of the Equation at the fifteenth Step, for as  $-aax - aax$  produces  $aaaa$ , as well as  $axaa$ , therefore in the Extraction of such Roots, it is doubtful whether the Root is  $-aa$ , or  $aa$ , let us now make a new Extraction, and suppose it to be  $-aa$ .

14 w 2	21	$-aa + ax - xx + bb = \sqrt{bbbb - xxbb}$ $= b\sqrt{bb - xx}$
--------	----	--

Having supposed the Root of  $aaaa$  the first pure Power to be  $-aa$ , I go to the next pure Power, which is  $aaxx$ , whose Root is  $ax$ ; but to determine its Sign, observe the Sign of the double Rectangle of these two Roots, viz. of  $2aaxx$ , which being  $-$ , I therefore make it  $+ax$ , as  $-$  into  $+$  produces  $-$ .

The next pure Power is  $xxxx$ , whose Root is  $xx$ , then observe the Sign of the double Rectangle of this, and either of the two former Roots, as of  $ax$ , now the Sign of  $2axxx$  is  $-$ , therefore in the Root make it  $-xx$ , for  $+ax \times -xx$  produces  $-axxx$ .

The last pure Power is  $bbbb$ , whose Root is  $bb$ , and observe the Sign of the double Rectangle of this, and either of the other Roots, as suppose the last, the double Rectangle of these two

Roots is  $2bbxx$ , which being —, therefore make it  $+bb$ , as  $-xx \times bb$  gives  $-xxbb$ .

Now transpose  $aa$ , it being *negative*.

$$\begin{array}{l|l} 21 + aa & 22 \\ 22 - b\sqrt{bb-xx} & 23 \\ 23 - ax & 24 \end{array} \left| \begin{array}{l} aa + b\sqrt{bb-xx} = ax - xx + bb \\ aa = bb + ax - xx - b\sqrt{bb-xx} \\ aa - ax = bb - xx - b\sqrt{bb-xx} \end{array} \right.$$

Here the Equation is *quadratic*, and because  $-xx - b\sqrt{bb-xx}$  is greater than  $bb$ , it is therefore *ambiguous*.

$$\begin{array}{l|l} 24 c \square & 25 \\ 25 w 2 & 26 \\ 26 + \frac{x}{2} & 27 \end{array} \left| \begin{array}{l} aa - ax + \frac{xx}{4} = bb - \frac{3xx}{4} - b\sqrt{bb-xx} \\ a - \frac{x}{2} = \sqrt{bb - \frac{3xx}{4}} - b\sqrt{bb-xx} \\ a = \frac{x}{2} \pm \sqrt{bb - \frac{3xx}{4}} - b\sqrt{bb-xx} \end{array} \right.$$

$$\begin{array}{r} x = 14 \\ x = 14 \\ \hline 56 \\ 14 \\ \hline xx = 196 \end{array}$$

$$\begin{array}{r} b = 14.75 \\ b = 14.75 \\ \hline \end{array}$$

$$\begin{array}{r} 7375 \\ 10325 \\ 5900 \\ 1475 \\ \hline \end{array}$$

$$\begin{array}{r} 217.5625 = bb \\ - 196 = -xx \\ \hline \end{array}$$

$$21.5625 = (4.64 = \sqrt{bb-xx})$$

$$\begin{array}{r} 86) 556 \\ 516 \\ \hline \end{array}$$

$$\begin{array}{r} 924) 4025 \\ 3696 \\ \hline \end{array}$$

$$329$$

$$14.75 = b \quad xx = 196$$

$$4.64 = \sqrt{bb - xx}$$

$$\begin{array}{r} 5900 \\ 8850 \\ \hline 5900 \end{array}$$

$$\begin{array}{r} 3 \\ \hline 4) 588 = 3xx \end{array}$$

$$147 = \frac{3xx}{4}$$

$$68.4400 = b\sqrt{bb - xx}$$

$$147 = \frac{3xx}{4}$$

$$215.44 = b\sqrt{bb - xx} + \frac{3xx}{4}$$

$$217.5625 = bb$$

$$-215.44 = -\frac{3xx}{4} - b\sqrt{bb - xx}$$

$$2.1225 \text{ (1.46 nearest)} = \sqrt{bb - \frac{3xx}{4}} - b\sqrt{bb - xx}$$

$$\begin{array}{r} 1 \\ \hline 24) 112 \\ 96 \\ \hline 286) 1625 \end{array}$$

$$7 = \frac{x}{2}$$

$$\pm 1.46 = \sqrt{bb - \frac{3xx}{4}} - b\sqrt{bb - xx}$$

$$8.46 = a = \text{A C.}$$

$$\text{or } 5.54 = a = \text{A C.}$$

But if  $a = 8.46$  then by the ninth Step  $y = \frac{xx - xa + aa}{b}$

$$a = 8.46$$

$$x = 14.$$

$$\begin{array}{r} 3384 \\ 846 \\ \hline \end{array}$$

$$ax = 118.44$$

$$\begin{array}{r}
 8.46 = a \\
 8.46 = a \\
 \hline
 5076 \\
 3384 \\
 6768 \\
 \hline
 71.5716 = a a \\
 196 = x x \\
 \hline
 267.5716 = a a + x x \\
 - 118.44 = - a x \\
 \hline
 b = 14.75) 149.1316 (10.11 = y = A B. \\
 \hline
 1475 \\
 \hline
 1631 \\
 1475 \\
 \hline
 1566 \\
 1475 \\
 \hline
 91
 \end{array}$$

Because  $a = A C = 8.46$  therefore  $B C = x - a = 5.54$ .

That these are the three Sides of a right-angled Triangle may be tried, by squaring and adding them, to see if they agree with the Property of the Figure.

5.54	10.11	8.46
5.54	10.11	8.46
<hr/> 2216	<hr/> 1011	<hr/> 5076
2770	1011	3384
2770	10110	6768
<hr/> 30.6916	<hr/> 102.2121	<hr/> 71.5716
71.5716		
<hr/> 102.2632		
102.2121		
<hr/>		

.0511 the Difference which arises from the Inaccuracy of the Fractions.

But if the last Process is too perplexing, the same Question may be done otherwise, thus,

Let



Let  $AC + BC = 14 = x$ , and  $a =$  the Difference between  $AC$  and  $BC$ , whence, as in the former Questions, the greater Leg or  $AC = \frac{x+a}{2}$ , and the lesser Leg  $BC = \frac{x-a}{2}$ .

Again, put  $AB + CD = 14.75 = b$ , and the Difference between  $AB$  and  $CD = y$ , then for the Reasons already mentioned  $AB = \frac{b+y}{2}$ , and  $CD = \frac{b-y}{2}$ .

Now because the Triangle  $ACB$  is right-angled,

by 47 e. 1	1	$\frac{xx + 2xa + aa}{4} + \frac{xx - 2xa + aa}{4}$
		$= \frac{bb + 2by + yy}{4}$

Because the Triangles  $ACB$  and  $BCD$  are similar, therefore

by 4 e. 6	2	$AB : AC :: BC : CD$
in Symbols	3	$\frac{b+y}{2} : \frac{x+a}{2} :: \frac{x-a}{2} : \frac{b-y}{2}$
3 ..	4	$\frac{bb-yy}{4} = \frac{xx-aa}{4}$ or $bb-yy=xx-aa$
from the first	5	$\frac{xx+aa}{2} = \frac{bb+2by+yy}{4}$

The Question being contained in the fourth and fifth Equations, and there being no other Powers of  $a$  but  $aa$  in both those Equations, exterminate that unknown Quantity.

$4 \div$	6	$aa = xx + yy - bb$
$5 \times 4$	7	$2xx + 2aa = bb + 2by + yy$
$7 - 2xx$	8	$2aa = bb + 2by + yy - 2xx$
$8 \div 2$	9	$aa = \frac{bb + 2by + yy - 2xx}{2}$
$6 \cdot 9$	10	$xx + yy - bb = \frac{bb + 2by + yy - 2xx}{2}$
$10 \times 2$	11	$2xx + 2yy - 2bb = bb + 2by + yy - 2xx$
$11 - yy$	12	$2xx + yy - 2bb = bb + 2by - 2xx$
$12 + 2bb$	13	$2xx + yy = 3bb + 2by - 2xx$
$13 - 2xx$	14	$yy = 3bb + 2by - 4xx$
$14 - 2by$	15	$yy - 2by = 3bb - 4xx$

I

Here

Here the Equation is *quadratic*, and since  $-4xx$  is greater than  $3bb$  it is *ambiguous*.

$$\begin{array}{r|l}
 15c \square & 16 \left| \begin{array}{l} yy - 2by + bb = bb + 3bb - 4xx \\ \quad \quad \quad = 4bb - 4xx \end{array} \right. \\
 16w2 & 17 \left| \begin{array}{l} y - b = \sqrt{4bb - 4xx} = 2\sqrt{bb - xx} \\ 17. + b \end{array} \right. \\
 & 18 \left| \begin{array}{l} y = b \pm 2\sqrt{bb - xx} = 14.75 \pm 9.3 \\ \quad \quad \quad = 24.05 \text{ or } 5.45 \end{array} \right.
 \end{array}$$

But  $y$  cannot be 24.05 for the Sum of the Legs is only 14.75 therefore  $y = 5.45$

then by Step 6th | 19 |  $a = \sqrt{xx + yy - bb} = 2.85$

Then  $AB = \frac{b+y}{2} = 10.1$   $AC = \frac{x+a}{2} = 8.42$   $BC = \frac{x-a}{2} = 5.57$  which three Numbers nearly agree with the

Property of the right-angled Triangle, but not exactly, because of the Imperfection of the Fractions.

The Reader may observe, that in several of the Geometrical Questions, after Letters are put for one or more of the unknown Quantities, we then get Expressions for the other Parts of the Figure from its Properties, and therefore avoid using a greater Number of unknown Quantities, and in general the Solution of Questions are more neat and elegant, the fewer unknown Quantities are used in the Work.

*The Method of resolving Questions, which contain four Equations, and four unknown Quantities.*

72. **W**HEN the Question contains four Equations, and there are four unknown Quantities in each Equation; find the Value of one of the unknown Quantities in one of the given Equations, and for that unknown Quantity in the other three Equations write this Value of it, which then reduces the Question to three Equations, and three unknown Quantities.

Then

Then find the Value of one of these three unknown Quantities in one of these three Equations, and for that unknown Quantity in the other two Equations write this Value of it, which reduces the Question to two Equations, and two unknown Quantities.

Then find the Value of one of the unknown Quantities in each of these two Equations, and making these Equations equal to one another, we shall have an Equation with only one unknown Quantity, which being reduced, will answer the Question.

Question 105. *A Father gave 1000l. to his four Sons A, B, C, D.*

*If A's Share was added to twice B's Share, from which Sum subtracting twice C's and D's Shares, there remains 650 Pounds:*

*And if from A's Share there is subtracted three times B's Share, to the Remainder adding twice C's Share, from which Sum subtracting five times D's Share, there remains 400 Pounds:*

*But if to A's Share there is added four times B's Share, from which Sum subtracting three times C's Share, and to the Remainder adding six times D's Share, the Sum is 1150 Pounds. How much had each Son?*

Let  $a = A$ 's Share,  $e = B$ 's Share,  $y = C$ 's Share,  $u = D$ 's Share,  $s = 1000$ ,  $m = 650$ ,  $n = 400$ ,  $b = 1150$ .

	1	$a + e + y + u = s$	} By the Question.
	2	$a + 2e - 2y - 2u = m$	
	3	$a - 3e + 2y - 5u = n$	
	4	$a + 4e - 3y + 6u = b$	
from the first	5	$a = s - u - y - e$	
5 . 2	6	$s + e - 3y - 3u = m$	
5 . 3	7	$s - 4e + y - 6u = n$	
5 . 4	8	$s + 3e - 4y + 5u = b$	

Here the Question is reduced to three Equations, and three unknown Quantities.

from the sixth	9	$e = m + 3u + 3y - s$
9 . 7	10	$s - 4m - 12u - 12y + 4s + y - 6u = n$
9 . 8	11	$s + 3m + 9u + 9y - 3s - 4y + 5u = b$

Here the Question is reduced to two Equations, and two unknown Quantities.

10 contracted	12	$5s - 4m - 18u - 11y = n$
11 contracted	13	$-2s + 3m + 14u + 5y = b$
from the twelfth	14	$y = \frac{5s - 4m - 18u - n}{11}$
from the thirteenth	15	$y = \frac{b + 2s - 3m - 14u}{5}$
14 . 15	16	$\frac{b + 2s - 3m - 14u}{5} = \frac{5s - 4m - 18u - n}{11}$
16 x	17	$11b + 22s + 33m - 154u = 25s - 20m - 90u - 5n$
17 ±	18	$64u = 11b - 3s - 13m + 5n$
18 ÷ 64	19	$u = \frac{11b - 3s - 13m + 5n}{64} = 50, \text{ the Share of D.}$
then by Step 15th	20	$y = \frac{b + 2s - 3m - 14u}{5} = 100, \text{ the Share of C.}$
and by Step 9th	21	$e = m + 3u + 3y - s = 100, \text{ the Share of B.}$
and by Step 5th	22	$a = s - u - y - e = 750, \text{ the Share of A.}$

And in the same Manner may any other Question in the like Circumstances be answered.

I shall now add a few Questions of a different Nature, and such as are generally first proposed to Learners, but as they require a little more Sagacity to express their Conditions, have hitherto been avoided, imagining the Learner is more perplexed to express, or find out the Equations resulting from such Questions, than to resolve the Equations; and therefore they were thought not so proper at the Beginning of this Work.

*Question 106. A Person bought two Horses A and B, which with the Trappings cost 100 Pounds:*

*Now if the Trappings were laid on the Horse A, both Horses were of equal Value:*

*But if the Trappings be laid on the Horse B, he will be double the Value of the Horse A. How much did each Horse cost?*

Let  $b = 100$ ,  $a =$  the Value of the Horse B and Trappings, then  $b - a =$  the Value of the Horse A.

Now because the Horse B and Trappings are double the Value of the Horse A,

hence ,

$$\begin{array}{l|l|l} \text{hence} & 1 & a = 2b - 2a \text{ by the Question.} \\ 1 + 2a & 2 & 3a = 2b \\ 2 \div 3 & 3 & a = \frac{2b}{3} = \frac{200}{3} = 66 \frac{2}{3} \text{ Pounds, the} \end{array}$$

Price of the Horse B and Trappings. Consequently  $100 - 66 \frac{2}{3}$

$= 33 \frac{1}{3}$  Pounds, the Price of the Horse A.

But to find what the Trappings cost, and by that Means to find the Price of the Horse B, let  $y$  = the Price of the Trappings.

Now the Trappings taken from the Horse B, and laid upon the Horse A, both Horses being then of equal Value,

$$\begin{array}{l|l|l} \text{therefore} & 1 & 33 \frac{1}{3} + y = 66 \frac{2}{3} - y \\ 1 + y & 2 & 33 \frac{1}{3} + 2y = 66 \frac{2}{3} \\ 2 - 33 \frac{1}{3} & 3 & 2y = 33 \frac{1}{3} \\ 3 \div 2 & 4 & y = 16 \frac{2}{3} \text{ Pounds, the Price of the} \\ & & \text{(Trappings.)} \end{array}$$

Consequently  $33 \frac{1}{3} + 16 \frac{2}{3} = 50$  Pounds, the Price of the Horse B.

Question 107. *A Labourer in 40 Weeks Labour saved 28 Crowns — the Pay of three Weeks, and found he had spent 36 Crowns + the Pay of eleven Weeks. How much did he receive a Week?*

Let  $a$  = his weekly Pay.

$$\begin{array}{rcll} \text{Then he had saved Crowns} & - & - & 28 - 3a \\ \text{And spent Crowns} & - & - & 36 + 11a \\ & & & \hline & & & 64 + 8a \end{array}$$

And as the Sum of these two must be equal to what he received for his forty Weeks Labour,

$$\begin{array}{l|l|l} \text{therefore} & 1 & 40a = 64 + 8a \\ 1 - 8a & 2 & 32a = 64 \\ 2 \div 32 & 3 & a = 2 \text{ Crowns, his weekly Pay.} \end{array}$$

Q q 2

Question

Question 108. *A Servant was hired for 12 Months, for which he was to have 24 Pounds with a Cloak; when he had served 8 Months he has Leave to go away, and instead of his Wages receives a Cloak and 13 Pounds. How much did the Cloak cost?*

Let  $a$  = the Price of the Cloak,  $b = 12$ ,  $d = 24$ ,  $m = 8$ ,  $x = 13$ .

Now  $d + a$  is what he did receive for serving eight Months.

And as the Pay for eight Months was proportional to what he was to receive for twelve Months, therefore,

therefore	2 +	1	$d + a : b :: x + a : m$
			When any four Quantities, or Numbers, are in Geometrical Proportion, the Product of the Extreams and Means are equal,
	3		$md + ma = bx + ba$
	4		$ba - ma = md - bx$
$3 \div b - m$			$a = \frac{md - bx}{b - m} = 9$ Pounds, the Price of the Cloak.

Question 109. *There is a Footman A, who goes 6 Miles a Day, and 8 Days after B follows him and goes 10 Miles a Day. In how many Days will B overtake A?*

Let  $b = 6$ ,  $d = 8$ ,  $m = 10$ ,  $a$  = the Number of Days B travels to overtake A, then as A begun to walk eight Days before B,

Hence the Number of Days that A travels, is  $- d + a$   
 And the Number of Miles A travels, is  $- bd + ba$   
 And the Number of Miles B travels, is  $- ma$

But when B overtakes A, they must have travelled an equal Number of Miles.

Therefore	1	1	$ma = bd + ba$
$1 - ba$	2		$ma - ba = bd$
$2 \div m - b$	3		$a = \frac{bd}{m - b} = 12$ , the Number of Days required, or the Time in which B will overtake A.

Question

Question 110. If a Scribe can in 8 Days write 15 Sheets, How many such Scribes can write 405 Sheets in 9 Days?

Let  $a$  = the Number of Scribes,  $b = 8$ ,  $d = 15$ ,  $m = 405$ ,  $n = 9$ .

Then	1	$b : d :: n : \frac{dn}{b}$ the Number of Sheets the Scribe can write in nine Days.
and	2	$\frac{dn}{b} : 1 :: m : \frac{bm}{dn}$ = the Number of Scribes to write the 405 Sheets in nine Days.
hence	3	$a = \frac{bm}{dn} = \frac{3240}{135} = 24$ , the Number (of Scribes required).

Question 111. A can do a Piece of Work once in 3 Weeks, B can do it three times in 8 Weeks, and C can do it five times in 12 Weeks. In how long Time can they do it jointly?

Let  $a$  = the Time required,  $b = 1$ ,  $d = 3$ ,  $g = 8$ ,  $n = 5$ ,  $m = 12$ , the Number 3 occurring twice, I put only  $d$  for it,

Then	1	$d : b :: a : \frac{ba}{d}$ , the Part of the Work that can be done by A in the Time sought.
and	2	$g : d :: a : \frac{da}{g}$ , the Part of the Work that can be done by B in the Time sought.
and	3	$m : n :: a : \frac{na}{m}$ , the Part of the Work that can be done by C in the Time sought.

And as these three Parts are to be equal to 1, or one Work,

therefore	4	$\frac{ba}{d} + \frac{da}{g} + \frac{na}{m} = 1$ .
whence	5	$a = \frac{1}{\frac{b}{d} + \frac{d}{g} + \frac{n}{m}} = \frac{1}{\frac{1}{3} + \frac{3}{8} + \frac{5}{12}}$ by

by reducing the Fractions  $\frac{1}{3} + \frac{3}{8} + \frac{5}{12}$  to a common Denominator, and adding and abbreviating them, we shall find  $\frac{1}{3} + \frac{3}{8} + \frac{5}{12} = \frac{9}{8}$ .

Whence  $a = \frac{1}{\frac{9}{8}} = \frac{8}{9}$  of a Week, by the Rule for Division (of Vulgar Fractions.

If the Week consists of 6 Days

$$\begin{array}{r} 8 \\ 9 \overline{) 48} \text{ (5 Days} \\ \underline{45} \end{array}$$

And the Days consist of  $\frac{3}{12}$  Hours

$$\begin{array}{r} 9 \overline{) 36} \text{ (4 Hours, that is, they will per-} \\ \underline{36} \text{ form the Work in five Days} \\ \text{four Hours.} \end{array}$$

Or the Equation  $\frac{ba}{g} + \frac{da}{g} + \frac{na}{m} = 1$ , may be reduced thus;

$$\begin{array}{l|l|l} 4 \times d & 6 & ba + \frac{dda}{g} + \frac{dna}{m} = d \\ 6 \times g & 7 & gba + dda + \frac{gdn a}{m} = gd \\ 7 \times m & 8 & mgba + mdda + gdn a = mgd \\ 8 \div & 9 & a = \frac{mgd}{mgb + mdd + gdn} = \frac{288}{324} \\ & & = \frac{8}{9} \text{ of a Week as above.} \end{array}$$

73. Having in this easy familiar Manner, by general and universal Rules, explained to the Learner the Elements of this celebrated Science, it may not be improper to raise his Curiosity, and animate him to exercise his Judgment in the Choice of Quantities for the Solution of the same Question, to give an Instance how much the Solution of Questions becomes more neat and elegant, by a judicious Choice of representing the unknown Quantities. The Question and its Solution is from the ingenious Mr. JOHN WARD's *Young Mathematician's Guide*.

Question



**Question 112.** *A Man playing at Hazard, or Dice, won the first Throw just so much Money as he had in his Pocket; the second Throw he won the square Root of what he then had, and five Shillings more; the third Throw he won the Square of all he then had; after which his whole Sum was 1121. 16 s. od. What Money had he when he began to play?*

Suppose	1	$a =$ his first Sum.
then	2	$2a =$ his Sum after the first Throw.
and	3	$\sqrt{2a} + 5 =$ his Winnings at the second Throw.
$2 + 3$	4	$\sqrt{2a} + 2a + 5 =$ his Sum after the second Throw.
$4 \odot 2$	5	$2a + 4a\sqrt{2a} + 10\sqrt{2a} + 4aa + 20a + 25 =$ his Winnings at the third Throw.
$4 + 5$	6	$24a + 4a\sqrt{2a} + 11\sqrt{2a} + 4aa + 30 = 2256$ Shillings.

Now to avoid these *furd Quantities*, let us make a second Supposition; thus,

Let	1	$2aa =$ his first Sum.
then	2	$4aa =$ his Sum after the first Throw.
and	3	$2a + 5 =$ his Winnings at the second Throw.
$2 + 3$	4	$4aa + 2a + 5 =$ his Sum after the second Throw.
$4 \odot 2$	5	$16aaaa + 16aaa + 40aa + 20a + 4aa + 25 =$ his Winnings at the third Throw.
$4 + 5$	6	$16aaaa + 16aaa + 48aa + 22a + 30 = 2256$

But to avoid these high Equations, let us make a third Supposition; thus,

Let	1	$\frac{aa}{2} =$ his first Sum.
then	2	$aa =$ his Sum after the first Throw.
and	3	$a + 5 =$ his Winnings at the second Throw.
$2 + 3$	4	$aa + a + 5 =$ his Sum after the second Throw.

But

But as it was the Square of  $aa + a + 5$  he won at the third Throw, to avoid the Trouble of squaring it,

substitute	5	$e = aa + a + 5$
then	6	$ee =$ his Winnings at the third Throw, consequently,
$5 + 6$	7	$ee + e = 2256$ Shillings
$7c \square$	8	$ee + e + 0.25 = 2256.25$
$8 \text{ as } 2$	9	$e + 0.5 = 47.5$
$9 - 0.5$	10	$e = 47$ . Because at the fifth Step $e$ was substituted for $aa + a + 5$
$5 \cdot 10$	11	$aa + a + 5 = 47$
$11 - 5$	12	$aa + a = 42$
$12c \square$	13	$aa + a + 0.25 = 42.25$
$13 \text{ as } 2$	14	$a + 0.5 = 6.5$
$14 - 0.5$	15	$a = 6$
whence	16	$\frac{aa}{2} = 18$ Shillings, the Money he had (when he first began to play.

The Learner will easily observe, that the third Solution is more neat and elegant than either of the other two; tho' I know of no *general* Rule that is given for the Choice of the Quantities to state the Question, but it is left to the Judgment and Sagacity of the Reader, and as such Methods must be attended with particular Difficulties to a Learner, I have avoided the perplexing him with them; but as he has now a general Method of solving Equations, he may exercise his Judgment at his own Discretion, in the Choice of different Quantities to represent the same Question.

*The Method of expressing the Power of any Quantity, by placing a Figure over it.*

74. **T**HERE is a more compendious Method of expressing the high Powers of any Quantity, than writing them at length, by placing a Figure over the Quantity thus,  $a^4$  is  $aaaa$ , and  $a^3$  is  $aaa$ , and  $a^1$  is  $a$ , and  $a^2b$  is  $aabbb$ , that is, the Figure that stands over the Letter shows to what Power

Of expressing the Power of any Quantity. 305

Power that Letter, or Quantity, is involved, which Method of Notation is generally used when the Powers are high. The Figures placed over the Quantity are called *Exponents*. The Mind being a little accustomed to this Method of Notation, will as easily manage an *Algebraic* Process, when the Powers are expressed by *Exponents*, as if they were repeated at length; and for the further Ease of the Learner, in this Method of Notation, we will resume the Solution of Question 90, expressing the Powers by *Exponents*, that the Learner may compare both the Operations together.

	1	$a^2 + a - e^2 = m$	} to find $a$ and $e$ .
	2	$e^2 + e = 2a$	
$1 + e^2$	3	$a^2 + a = m + e^2$	
$3 - m$	4	$a^2 + a - m = e^2$	
$2 - e$	5	$e^2 = 2a - e$	
$4 \cdot 5$	6	$a^2 + a - m = 2a - e$	
$6 + e$	7	$a^2 + a - m + e = 2a$	
$7 - a$	8	$a^2 - m + e = a$	
$8 + m$	9	$a^2 + e = a + m$	
$9 - a^2$	10	$e = a + m - a^2$	
$10 \odot 2$	11	$e^2 = a^2 + 2am + m^2 - 2a^2 - 2ma^2 + a^4$	
$2 \cdot 11 \cdot 10$	12	$a^2 + 2am + m^2 - 2a^2 - 2ma^2 + a^4 + a + m - a^2 = 2a$	
12 in Numbers	13	$188a + 8836 - 2a^3 - 188a^2 + a^4 + a + 94 = 2a$	
13 contracted	14	$187a + 8930 - 2a^3 - 188a^2 + a^4 = 0$	
$13 - a^4$	15	$-a^4 = 187a + 8930 - 2a^3 - 188a^2$	
$15 + 2a^3$	16	$-a^4 + 2a^3 = 187a + 8930 - 188a^2$	
$16 + 188a^2$	17	$-a^4 + 2a^3 + 188a^2 = 187a + 8930$	
$17 - 187a$	18	$-a^4 + 2a^3 + 188a^2 - 187a = 8930$	

In the same Manner the Learner may attempt the Solution of any of the other Questions, expressing the Powers by *Exponents*: One Thing is to be carefully observed, that the *Exponent* belongs only to the Letter which stands under it, and when it is only *Unity*, or 1, it is never set down, like the Co-efficient when it is *Unity* only, it is generally omitted in the Expression.

*The Method of knowing if a Question is limited, or admits but of one Answer; or if it is indetermined, that is, admits of several Answers.*

75. **T**HE Question being stated, that is, all the Equations being expressed which are necessary for the Solution of the Question, then if there are *more* unknown Quantities than there are *Equations*, the Question admits of a *Variety* of Answers, and is therefore unlimited or indetermined, *ex. gr.*

$$\left. \begin{array}{l} \text{Suppose } a + e = 40 \\ \text{And } e + y = 20 \end{array} \right\} \text{ to find } a, e, \text{ and } y.$$

Here there are three unknown Quantities, and only two Equations.

Now *e* being in both the given Equations, you may suppose it any Number under 20, the least of the two given Numbers, as for Example suppose *e* = 16.

Then the first Equation is  $a + 16 = 40$ .

And the second Equation is  $16 + y = 20$ .

From whence it will be easy to find *a* and *y*, but if *e* is supposed any other Number under 20, then there will be found different Numbers for *a* and *y*, and the like of any other Question, where the Number of unknown Quantities, are more than the Equations which arise from the Question.

But when the Number of given Equations are *just as many as the unknown Quantities required to be found*, then the Question generally admits but of *one* Answer, for then each of the Quantities sought hath generally but *one* single Value, thus as at Question 80, where we have

$$\left| \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right| \begin{array}{l} a + e + y = b = 18 \\ a + 3e - 2y = m = 9 \\ a + 4y - 2e = p = 21 \end{array}$$

Where *a* = 5, *e* = 6, and *y* = 7.

But

## To extract the Cube Root.

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But when the Number of given Equations exceeds the Number of Quantities sought, they not only limit the Question, but often render it impossible, as one of the Equations may be inconsistent with another; as for Example,

$$\left. \begin{array}{l|l} 1 & a + e = 16 \\ 2 & ae = 48 \\ 3 & a - e = 22 \end{array} \right\} \text{to find } a \text{ and } e.$$

Now here are three Equations, and but two unknown Quantities, and the first and second Equations include a possible Case, and it may be found what the Numbers are.

And if we take the second and third Equations, they likewise include a possible Case, for it may be determined what those Numbers are.

But all three Equations together render the Case impossible, the first Equation being incompatible with the third, as the Sum of two Numbers cannot be less than their Difference.

## To raise or invent a Method to extract the Cube Root.

76. **T**HIS is no more than the Method of *Converging Series* applied to the Solution of an Equation, one Side of which is the unknown Quantity, and is a pure Cube, or raised to the third Power only, *ex. gr.*

Suppose  $aaa = 9261$ , where 9261 is a Cube Number, now to find what  $a$ , or the Number is that being cubed will produce 9261, is to extract the Cube Root of 9261.

By the common Method of distinguishing of how many Places the Root will consist, by placing a Point over the Place of Units, and another over every third Figure, the Root will consist of two Places, therefore suppose the Cube Root to be

$$\begin{array}{r} 20 \\ 20 \\ \hline 400 \\ 20 \\ \hline \end{array}$$

8000 which being less than 9261 the given Number, the Cube Root of 9261 must be more than 20.

Now put  $r = 20$ , and  $e$  for what 20 wants of the true Root, then is  $r + e = a$ , or the Cube Root of 9261, and proceed as in the Method of *Converging Series*, Case 1. Page 230.

$$\text{If } | \ 1 \ | \ r + e = a,$$

Raise this Equation to the third Power, because it is the Cube Root, which is to be extracted.

$$\begin{array}{r|l} 1 \text{ @ } 3 & 2 \ | \ rrr + 3rre + 3ree + eee = aaa \\ \text{but} & 3 \ | \ aaa = 9261 \text{ by the Example,} \\ 2 \cdot 3 & 4 \ | \ rrr + 3rre + 3ree + eee = 9261 \end{array}$$

Put this Equation into Numbers, and reject all the Powers of  $e$  above  $ee$ , as in the Method of *Converging Series*.

$$\begin{array}{r|l} 4 \text{ in Numbers} & 5 \ | \ 8000 + 1200e + 60ee = 9261 \\ & \text{Because 8000 is less than 9261, trans-} \\ & \text{pose 8000} \\ 5 - 8000 & 6 \ | \ 1200e + 60ee = 1261 \\ & \text{Dividing by the Co-efficient of } ee. \\ 6 \div 60 & 7 \ | \ 20e + ee = 21.01 \\ & \text{Dividing by } 20 + e, \text{ that is, by the Co-} \\ & \text{efficient of } e \text{ plus } e, \text{ as in the Method} \\ & \text{of } \textit{Converging Series}, \\ 7 \div 20 + e & 8 \ | \ e = \frac{21.01}{20 + e} \end{array}$$

Operation in Numbers,

$$\begin{array}{r} 20) \ 21.01 \ (1 = e \\ + e = 1 \\ \hline \text{Divisor } 21 \quad 21 \\ \hline .01 \text{ Remainder rejected.} \end{array}$$

$$\begin{array}{r} r = 20 \\ + e = 1 \\ \hline \end{array}$$

$r + e = 21 = a$ , which being tried will be found to be the Cube Root of 9261. And by the same Method may the Cube Root of any other Number be extracted.

But to save the Trouble of repeating this Operation, when any Cube Root is to be extracted, the above Process may be made more general, by not turning the Equation at the fourth Step into Numbers, and putting any Letter for the given Number, whose Cube Root is to be extracted.

Suppose

Suppose as before  $aaa = 9261$ , let  $b = 9261$ .

Then  $aaa = b$ , to find  $a$ , or to extract the Cube Root.

Now make a Supposition that 20 is the Root, which being tried as before, it will be found too little. Then put  $20 = r$ , and because 20 is too little,  $r$  in this Case is usually called *less than just*; and for what  $r$  wants of the true Root put  $e$ , whence  $r + e$  will be the true Root, or equal to  $a$ .

Hence	1	$r + e = a$ Raise this Equation to the third Power as before.
1 $\odot$ 3	2	$rrr + 3rre + 3ree + eee = aaa$
but	3	$aaa = b$ as above
2 $\cdot$ 3	4	$rrr + 3rre + 3ree + eee = b$

As we know  $rrr$  to be less than  $b$ , by finding the Cube of 20 was less than the given Number, therefore transpose  $rrr$ , and reject the Powers of  $e$  above  $ee$ .

4 $-$ $rrr$	5	$3rre + 3ree = b - rrr$ Dividing by the Co-efficient of $ee$ ,
5 $\div$ $3r$	6	$re + ee = \frac{b - rrr}{3r}$

As there will be another Division before the Operation is finished, to keep the Fraction as simple as may be, substitute

$$D = \frac{b - rrr}{3r}.$$

Then  $7 \mid re + ee = D$

Now dividing by  $r + e$ , that is, the Co-efficient of  $e$  plus  $e$ ,

$$7 \div r + e \mid 8 \mid e = \frac{D}{r + e}, \text{ THEOREM 1.}$$

Operation in Numbers,

$$\begin{array}{r}
 b = 9261 \\
 - rrr = - 8000 \\
 \hline
 3r = 60 \overline{) 1261} \quad (21.01 = D. \\
 \underline{120} \phantom{00} \\
 61 \phantom{00} \\
 \underline{60} \phantom{00} \\
 100 \phantom{00} \\
 \underline{60} \phantom{00} \\
 40
 \end{array}$$

$$r = 20.$$

$$\begin{array}{r}
 r = 20) 21.01 = D \ (1 = e. \\
 + e = \underline{1} \\
 \text{Divisor } 21 \ 21 \\
 \hline
 .01 \text{ Remainder neglected.}
 \end{array}$$

$$\begin{array}{r}
 r = 20 \\
 + e = \underline{1} \\
 r + e = 21 = a, \text{ the Cube Root required as before.}
 \end{array}$$

Now suppose it was required to extract the Cube Root of 132651.

Here according to the Method of pointing, the Root will consist of two Places, and to make a tolerable near Supposition at the first Trial, the first Period being 132, I consider what whole Number cubed will be the nearest to 132, and I find it to be 5, then as the Root consists of two Places, I supply the next Place with a *Cypher*, and suppose the Root to be 50, which I know is less than the true Root, as the Cube of 5 is less than 132.

Hence, as before, we are to determine what the Number is, that 50 wants of the true Root of 132651.

Then putting  $r = 50$ , and  $e$  what it wants of the true Root, and  $b = 132651$ , we have just the same substituted Letters as in the last Example; and if the Operation was repeated it will be exactly the same, it is therefore needless to repeat the Work, but only observing the Equation, or *Theorem* to find  $e$ , which is  $e = \frac{D}{r + e}$ , and by Substitution we have  $D = \frac{b - rrr}{3r}$ .

$$\begin{array}{r}
 \text{Now } b = 132651 \\
 - rrr = - 125000 \\
 \hline
 3r = 150) 7651 \ (51.006 = D. \\
 \underline{750} \\
 151 \\
 \underline{150} \\
 1000 \\
 \underline{900} \\
 100
 \end{array}$$

$$r = 50$$



$$\begin{array}{r}
 r = 50) \ 51.006 = D \ (1 = e \\
 + e = \frac{1}{51} \\
 \hline
 \text{Divisor } 51 \quad 51 \\
 \hline
 .006 \text{ Remainder neglected.}
 \end{array}$$

Now it was  $r = 50$   
 We have found  $e = \frac{1}{51}$

$r + e = 51$  the Cube Root of 132651, which being tried will be found to be true.

And in the same Manner, the Cube Root of any other Number may be extracted, without repeating the *Algebraic Work*, when the Number assumed for the Root is less than the true Root: But when the Number assumed for the Root is too much, or more than the true Root, then we proceed as in the following Example, in the same Manner as at the second *Case of Converging Series*, Page 235.

Required to extract the Cube Root of 24389, or  $aaa = 24389$ .

By the usual Method of pointing, the Root will consist of two Figures, the first Period of the given Number is 24, and the Cube of 3 being the nearest of whole Numbers to 24, and supplying the other Place of the Root with a *Cypher*, I suppose 30 to be the Cube Root of 24389, but the Cube of 30 is 27000, which being more than the given Number, the Cube Root cannot be so much as 30.

Therefore let  $r = 30$ , which is now too great or *more than just*, and what 30 is too much call  $e$ , then will  $r - e = a$ , or the true Cube Root required, and calling the given Number  $24389 = b$ ,

we have	1	$r - e = a$ Raife this Equation to the third Power as before.
1 - 3	2	$rrr - 3rre + 3ree - eee = aaa$
but	3	$aaa = 24389 = b$ , as $b$ is put for the given Number.
2 . 3	4	$rrr - 3rre + 3ree - eee = b$

Because  $b$  is less than  $rrr$ , transpose  $b$  and reject the Powers of  $e$  above  $ee$ .

4 - b	5	$rrr - b - 3rre + 3ree = 0$ , for one Side of the Equation subtracted from the other Side must leave 0, or <i>nothing</i> .
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I

Then

Then transpose all the Powers of  $e$ , to the other Side of the Equation.

$$\begin{array}{r|l} 5 + 3rre & 6 \\ 6 - 3ree & 7 \\ 7 \div 3r & 8 \end{array} \left| \begin{array}{l} 3rre = rrr - b + 3ree \\ 3rre - 3ree = rrr - b \\ \text{Dividing by the Co-efficient of } ee, \\ re - ee = \frac{rrr - b}{3r} \end{array} \right.$$

As there will be another Division before the Operation is finished, to keep the Fraction as simple as may be, substitute

$$G = \frac{rrr - b}{3r}.$$

$$\text{Then } | 9 | re - ee = G$$

Now dividing by  $r - e$ , that is, by the Co-efficient of  $e$  minus  $e$ ,

$$9 \div r - e \mid 10 \mid e = \frac{G}{r - e}, \text{ THEOREM 2.}$$

Operation,

$$\begin{array}{r} rrr = 27000 \\ - b = - 24389 \\ \hline 3r = 90 \quad 2611 \quad (29.01 = G. \\ \quad \quad \quad 180 \\ \quad \quad \quad \hline \quad \quad \quad 811 \\ \quad \quad \quad 810 \\ \quad \quad \quad \hline \quad \quad \quad 100 \\ \quad \quad \quad 90 \\ \quad \quad \quad \hline \quad \quad \quad 10 \end{array}$$

$$\begin{array}{r} r = 30 \quad 29.01 = G \quad (1 = e \\ - 1 \\ \hline \text{Divisor } 29 \quad 29 \quad \quad \quad .01 \text{ Remainder neglected.} \end{array}$$

$r = 30$   
 $- e = - 1$   
 $r - e = 29 = a$ , which being cubed will be found the true  
 Cube Root of 24389.

In

In this Case, the Quotient Figure is subtracted from the Divisor as it is found, the Divisor at the tenth Step being  $r - e$ , whereas in *Theorem 1*, p. 309, it was at the eighth Step  $r + e$ .

Now as the first supposed Root must be too great or too little, unless it happens to be taken exact at the first Time, therefore these two *Theorems* will extract the Cube Root of any Number, as in the following Example.

Let it be required to extract the Cube Root of 14526.784

From pointing the whole Numbers according to the usual Method in common Arithmetic, the Root will consist of two Places of Integers, the first Period of the given Number being 14, the Cube of the whole Number which is nearest to 14 is 2; and supplying the other Place of the Root with a *Cypher*, I suppose the Root of the given Number to be 20, which is too little, or less than just, the Cube of 2 the first Figure in the Root being less than 14, the first Period in the given Number.

Then putting  $b = 14526.784$   $r = 20$ , and  $e$  what 20 wants of the true Root, we proceed as at *Theorem 1*, page 309, where  $e = \frac{D}{r+e}$ , and by Substitution  $D = \frac{b - rrr}{3r}$ .

$$\begin{array}{r}
 b = 14526.784 \\
 - rrr = - 8000. \\
 \hline
 3r = 60 \quad 6526.784 \quad (108.78 \text{ nearest} = D. \\
 \quad \quad \quad 60 \\
 \quad \quad \quad \hline
 \quad \quad \quad 526 \\
 \quad \quad \quad 480 \\
 \quad \quad \quad \hline
 \quad \quad \quad 467 \\
 \quad \quad \quad 420 \\
 \quad \quad \quad \hline
 \quad \quad \quad 478
 \end{array}$$

$$\begin{array}{r}
 r = 20 \quad 108.78 = D \quad (4.44 = e. \\
 + e = 4 \\
 \hline
 \text{Divisor } 24 \quad 96 \\
 \quad \quad \quad 4.4 \quad 1278 \\
 \hline
 \text{Divisor } 28.4 \quad 1136 \\
 \quad \quad \quad .44 \quad 14200 \\
 \hline
 \text{Divisor } 28.84 \quad 11536 \\
 \quad \quad \quad \quad 2664
 \end{array}$$

The Reader will observe that the Quotient Figure is added twice to the Divisor to compleat it, in the same Manner as at the Method of *Converging Series*, Page 233.

$$\text{Now } r = 20$$

$$+ e = 4.44$$

$r + e = 24.44$  and to try whether this is the true Root of the given Number cube it.

$$\begin{array}{r}
 24.44 \\
 24.44 \\
 \hline
 9776 \\
 9776 \\
 9776 \\
 4888 \\
 \hline
 597.3136 \\
 24.44 \\
 \hline
 23892544 \\
 23892544 \\
 23892544 \\
 11946272 \\
 \hline
 \end{array}$$

14598.344384 which being greater than the given Number, the Root cannot be so much as 24.44

To approach still nearer to the true Root, make a second Operation, supposing the Number last found, viz. 24.44 to be  $r$ , and put  $e$  for what that Number is too much, then  $r - e$  will be the true Root, and putting the given Number 14526.784 =  $b$ , we proceed as at *Theorem 2*, Page 312, where  $e = \frac{G}{r - e}$ ,

and by Substitution  $G = \frac{rrr - b}{3r}$ .

$$\begin{array}{r}
 rrr = 14598.344384 \\
 - b = - 14526.784 \\
 \hline
 3r = 73.32 \quad 71.560384 \quad (.976 = G. \\
 65988 \\
 \hline
 55723 \\
 51724 \\
 \hline
 43998 \\
 43992 \\
 \hline
 6
 \end{array}$$

$$\begin{array}{r}
 r = 24.44) .9760 = G (.04 = e. \\
 -e = - .04 \\
 \hline
 \text{Divisor } 24.40 \quad 9760 \\
 \hline
 0
 \end{array}$$

Now  $r = 24.44$  by the first Operation.

$$\begin{array}{r}
 -e = - .04 \\
 r - e = 24.4 \text{ which being cubed, will be found the true} \\
 \text{Root of } 14526.784
 \end{array}$$

Therefore by the second Operation the true Root is found.

For a further Variety, let it be again required to extract the Cube Root of the same Number 14526.784

But let us suppose the Cube Root to be 30, the Cube of which being 27000, the Root cannot be so much as 30, then putting  $r = 30$ , we shall have  $r$  too great or *more than just*, and putting  $e$  what it is too much, then  $r - e$  will be the true Root, and calling the given Number  $14526.784 = b$ , we proceed as at *Theorem 2*, Page 312, where  $e = \frac{G}{r - e}$ , and by

$$\text{Substitution } G = \frac{rrr - b}{3r}.$$

$$\begin{array}{r}
 rrr = 27000. \\
 -b = - 14526.784 \\
 \hline
 3r = 90) 12473.216 \quad (138.591 = G. \\
 \hline
 90
 \end{array}$$

$$\begin{array}{r}
 347 \\
 270 \\
 \hline
 773 \\
 720 \\
 \hline
 532 \\
 450 \\
 \hline
 821 \\
 810 \\
 \hline
 116 \\
 90 \\
 \hline
 26
 \end{array}$$

$$\begin{array}{r} r = 30) \quad 138.591 = G \quad (5.7 \\ - e = - 5 \end{array}$$

$$\text{Divisor } 25 \quad 125$$

$$\begin{array}{r} - 5.7 \quad 1359 \end{array}$$

$$\text{Divisor } 19.3 \quad 1351$$

$$8$$

$$\begin{array}{r} r = 30. \\ - e = - 5.7 \end{array}$$

$r - e = 24.3$  to try whether this is the Cube Root of  
14526.784 cube 24.3

$$\begin{array}{r} 24.3 \\ 24.3 \\ \hline 729 \\ 972 \\ 486 \\ \hline 592.49 \\ 24.3 \\ \hline 177147 \\ 236196 \\ 118098 \\ \hline \end{array}$$

14348.907 which being less than the given Number  
14526.784 the Cube Root must be  
more than 24.3

Now for a second Operation, and let  $r = 24.3$  and what it wants of the true Root call  $e$ , then will  $r + e$  be the true Root, and still calling the given Number  $14526.784 = b$ , we now proceed as at *Theorem 1*, Page 309, where  $e = \frac{D}{r + e}$ , and by Substitution  $D = \frac{b - rrr}{3r}$ .

$$b =$$

*To extract the Biquadrate Root.*

317

$$\begin{array}{r}
 b = 14526.784 \\
 - r r r = - 14348.907 \\
 \hline
 3 r = 72.9 \quad 177.877 \quad (2.44 = D. \\
 \quad \quad \quad 1458 \\
 \hline
 \quad \quad \quad 3207 \\
 \quad \quad \quad 2916 \\
 \hline
 \quad \quad \quad 2917 \\
 \quad \quad \quad 2916 \\
 \hline
 \quad \quad \quad \quad \quad \quad 1 \\
 r = 24.3) \quad 2.44 = D \quad (.1 = e. \\
 \quad \quad \quad + .1 \\
 \hline
 \text{Divisor } 24.4 \quad \underline{244} \\
 \quad \quad \quad \quad \quad \quad 0
 \end{array}$$

$r = 24.3$  by the first Operation.

$$+ e = .1$$

$r + e = 24.4$  the true Root as before.

In the same Manner may the Cube Root of any other Number be extracted, and tho' the true Root may not always be exactly had, yet by repeating the Operation you may approach to it, within any assignable Degree of Exactness, and if a small Mistake happens in the first, it will be corrected at the second Operation.

*To extract the Biquadrate, or fourth Root.*

THIS Operation proceeds in the same Manner as in the Cube Root, only raising the  $r + e$  or  $r - e$  to the fourth Power, thus,

Required the Biquadrate or fourth Root of 194481, or of  $a a a a = 194481$ .

By placing a Point over the Place of Units, and another over every fourth Figure, we shall find the Root will consist of two Figures: And the first Period of the given Number being 19, now the Biquadrate or fourth Power of 2 being 16, which being the nearest in Integers, and supplying the other Place of the Root with a Cypher, suppose 20 to be the Biquadrate Root of 194481; but the Biquadrate of 20 being only 160000, the Root must be more than 20. Now let  $r = 20$ , and putting  $e$  for what 20 wants of the true Root, then will  $r + e = a$  be the true Root required; and calling the given Number  $194481 = b$ , then  $a a a a = b$ .

Now

Now  $| 1 | r + e = a.$

Raise this Equation to the fourth Power, because it is the *Biquadrate* Root that is to be extracted.

$$\begin{array}{r|l|l}
 1 \text{ } \textcircled{4} & 2 & rrrr + 4rrre + 6rree = aaaa, \\
 & & \text{all the Powers of } e \text{ above } ee \text{ being} \\
 & & \text{rejected.} \\
 \text{But} & 3 & aaaa = b, b \text{ being put equal to the} \\
 & & \text{given Number.} \\
 2 \cdot 3 & 4 & rrrr + 4rrre + 6rree = b. \\
 & & \text{Because } rrrr \text{ is less than } b, \text{ transpose} \\
 & & rrrr. \\
 4 - rrrr & 5 & 4rrre + 6rree = b - rrrr \\
 & & \text{Dividing by the Co-efficient of } ee, \\
 5 \div 6rr & 6 & \frac{2re}{3} + ee = \frac{b - rrrr}{6rr}
 \end{array}$$

As there will be another Division before the Operation is finished, therefore as in the Cube Root, put  $D = \frac{b - rrrr}{6rr}.$

$$\text{Then } | 7 | \frac{2re}{3} + ee = D.$$

Now dividing by  $\frac{2r}{3} + e$ , that is, the Co-efficient of  $a$  plus  $e$ ,

$$8 \div \frac{2r}{3} + e \quad | \quad 8 \quad | \quad e = \frac{D}{\frac{2r}{3} + e} \quad \text{THEOREM 1.}$$

$$\begin{array}{r}
 \text{Operation } b = 194481 \\
 - rrrr = - 160000 \\
 \hline
 6rr = 2400 \quad 34481 \quad (14.367 = D. \\
 \quad \quad \quad 2400 \\
 \quad \quad \quad \hline
 \quad \quad \quad 10481 \\
 \quad \quad \quad 9600 \\
 \quad \quad \quad \hline
 \quad \quad \quad 8810 \\
 \quad \quad \quad 7200 \\
 \quad \quad \quad \hline
 \quad \quad \quad 16100 \\
 \quad \quad \quad 14400 \\
 \quad \quad \quad \hline
 \quad \quad \quad 17000
 \end{array}$$



To Extract the Biquadrate Root. 319

$$\frac{2r}{3} = 13.33) \quad 14.367 = D (1. = e.$$

$$+ 1.$$

$$\begin{array}{r} 14.33 \quad 1433 \\ \hline \end{array}$$

37 Remainder neglected.

$$r = 20$$

$$+ e = 1$$

$r + e = 21 = a$ , which being raised to the fourth Power, will be found to be the *Biquadrate* Root of the given Number.

And if we here take the first Root too great, or more than the Truth, the Operation is the same as raising the second *Theorem* for the Cube Root.

Suppose  $aaaa = 456976$ , to find the *Biquadrate* Root.

The Root being found to consist of two Figures as before, and the first Period in the given Number being 45, and the *Biquadrate* of 3 being 81, supplying the other Place of the Root with a *Cypher*, let us suppose 30 to be the Root, but the *Biquadrate* of 30 is 810000, which being more than 456976, the Root cannot be so much as 30.

Then putting  $r = 30$ , and  $e$  what 30 is too much, we have  $r - e = a$  the Root required; and putting  $b = 456976$ , we then have  $aaaa = b$ .

Now	1	$r - e = a$
		Raising this Equation to the fourth
		Power as before, and neglecting the
		Powers of $e$ above $ee$ .
1 $\ominus$ 4	2	$rrrr - 4rrre + 6rree = aaaa$
But	3	$aaaa = b$
2. 3	4	$rrrr - 4rrre + 6rree = b$

Because  $b$  is less than  $rrrr$  therefore transpose  $b$ .

4 - b	5	$rrrr - b - 4rrre + 6rree = 0$ ,
		one Side of the Equation being sub-
		tracted from the other, must leave
		nothing, now transpose the several
		Powers of $e$ .
5 + 4rrre	6	$4rrre = rrrr - b + 6rree$

6 -

$$\begin{array}{l|l}
 6 - 6rree & 7 \left| \begin{array}{l} 4rree - 6rree = rrrr - b \\ \text{Divide by the Co-efficient of } ee. \end{array} \right. \\
 7 \div 6rr & 8 \left| \begin{array}{l} \frac{2re}{3} - ee = \frac{rrrr - b}{6rr} \end{array} \right.
 \end{array}$$

For the same Reason as in the last Operation, substitute  
 $G = \frac{rrrr - b}{6rr}.$

$$\begin{array}{l|l}
 \text{Then} & 9 \left| \begin{array}{l} \frac{2re}{3} - ee = G \\ \text{Now divide by } \frac{2r}{3} - e, \text{ that is, by the} \\ \text{Co-efficient of } e \text{ left } e. \end{array} \right. \\
 9 \div \frac{2r}{3} - e & 10 \left| \begin{array}{l} e = \frac{G}{\frac{2r}{3} - e} \text{ THEOREM 2.} \end{array} \right.
 \end{array}$$

Operation,

$$\begin{array}{r}
 rrrr = 810000 \\
 - b = -456976 \\
 \hline
 6rr = 5400 \quad 353024 \quad (65.374 = G. \\
 \quad \quad \quad 32400 \\
 \quad \quad \quad \hline
 \quad \quad \quad 29024 \\
 \quad \quad \quad 27000 \\
 \quad \quad \quad \hline
 \quad \quad \quad 20240 \\
 \quad \quad \quad 16200 \\
 \quad \quad \quad \hline
 \quad \quad \quad 40400 \\
 \quad \quad \quad 37800 \\
 \quad \quad \quad \hline
 \quad \quad \quad 26000 \\
 \quad \quad \quad 21600 \\
 \quad \quad \quad \hline
 \quad \quad \quad 4400
 \end{array}$$

$$\frac{2r}{3} = 20. \quad 65.374 = G \quad (4.08 = e.$$

$$\begin{array}{r}
 \quad \quad \quad \underline{\quad 4 \quad} \\
 \text{Divisor } 16. \quad 64 \\
 \quad \quad \quad \underline{\quad .08 \quad} \quad 13740 \\
 \text{Divisor } 15.92 \quad 12736 \\
 \quad \quad \quad \hline
 \quad \quad \quad 1004
 \end{array}$$

$$r = 30$$

$$\begin{array}{r} r = 30 \\ - e = - 4.08 \\ \hline \end{array}$$

$r - e = 25.92 = a$ , and to try if this is the Root,  
raise 25.92 to the fourth Power.

$$\begin{array}{r} 25.92 \\ 25.92 \\ \hline 5184 \\ 23328 \\ 12960 \\ 5184 \\ \hline 671.8464 \\ 671.8464 \\ \hline 26873856 \\ 40310784 \\ 26873856 \\ 53747712 \\ 6718464 \\ 47029248 \\ 40310784 \\ \hline \end{array}$$

451377.58519296 which being less than the given Number  
456976, the true Root must be more  
than 25.92

Then for a second Operation let  $r = 25.92$  and for what it  
wants of the true Root put  $e$ , that now  $r + e = a$ , and still  
calling the given Number  $456976 = b$ , this is exactly the same  
Case as when we raised the first *Theorem*, Page 318, for the  
*Biquadrate* Root, whence we have no Occasion to repeat  
the *Algebraic* Work, but to use that *Theorem*, where

$$e = \frac{D}{\frac{2r}{3} + e}, \text{ and by Substitution } D = \frac{b - rrrr}{6rr}.$$

T t

b =

$$\begin{array}{r}
 b = 456976. \\
 -rrrr = -451377.5852 \\
 \hline
 6rr = 4031.0784) 5598.4148 \quad (1.3888 = D. \\
 \quad 40310784 \\
 \hline
 \quad 156733640 \\
 \quad 120932352 \\
 \hline
 \quad \quad 358012880 \\
 \quad \quad 322486272 \\
 \hline
 \quad \quad \quad 355266080 \\
 \quad \quad \quad 322486272 \\
 \hline
 \quad \quad \quad \quad 32779808
 \end{array}$$

$$\begin{array}{r}
 \frac{2r}{3} = 17.28) 1.3888 = D \quad (.08 = e. \\
 \quad + \quad .08 \\
 \hline
 \text{Divisor } 1736 \quad 13888 \\
 \hline
 \quad \quad 0
 \end{array}$$

$$\begin{array}{r}
 r = 25.92 \text{ by the first Operation.} \\
 + e = .08 \\
 \hline
 \end{array}$$

$r + e = 26. = a$ , which being involved to the fourth Power, will be found the true *Biquadrate* Root of 456976.

The Reader will easily observe that these two *Theorems* will extract the *Biquadrate* Root of any given Number, in the same Manner as the two *Theorems* did for the Cube Root.

In the same Method may *Theorems* be raised to extract any Root, it being no more than to suppose a Number to be the required Root, and try whether it is too great or too little; then calling it  $r + e$ , or  $r - e = a$ , or the true Root, as the Occasion requires, and raise this Equation as high as the Root is to be extracted, after which the Operation is the same as before.

*To turn Equations into Analogies.*

77. **S**UPPOSE there was given this Proportion  $a : b :: c : d$ ; then multiplying Extreams and Means we have this Equation  $ad = bc$ , now as we get an Equation from Quantities in continual Proportion, by multiplying the Extreams and Means, and making one Product equal to the other. Hence to turn any Equation into an Analogy, is only the reverse, by taking the Quantities that compose either Side of the Equation, and making them the two Extreams, and the Quantities that compose the other Side of the Equation, and making them the two Means in the Proportion.

To turn the Equation  $md = za$  into an Analogy.

One Side of the Equation is composed of the Quantities  $m$  and  $d$ .

And the other Side of the Equation is composed of the Quantities  $z$  and  $a$ .

Hence placing these Quantities according to the Direction, we have  $m : z :: a : d$

$$\begin{aligned} \text{Or } d : z :: a : m \\ \text{Or } z : d :: m : a \\ \text{Or } z : m :: d : a, \text{ \&c.} \end{aligned}$$

For multiplying the Extreams and Means of either of these Proportions, we shall still have the given Equation  $dm = za$ .

Again, suppose the Equation  $an = bdx$ , and it is required to find the Proportion of  $a$  to  $b$ .

Now one Side of the Equation is composed of the Quantities  $a$  and  $n$ .

And the other Side of the Equation is composed of the Quantities  $b$  and  $dx$ .

But in ranging these Quantities, make the Quantities  $a$  and whose Proportion is required, the first and second Terms in the Proportion, and place the other two Quantities so, that if the Extreams and Means were to be multiplied, they will produce the given Equation, and then we shall find  $a : b :: dx : n$ .

From the Equation  $dny = bxz$ , to find the Proportion of  $d$  to  $b$ .

By the Directions we shall find  $d : b :: xz : ny$ .

From  $\frac{a^n}{m} = p \times d$ , to find the Proportion of  $a$  to  $d$ , which is  $a : d :: p \times : \frac{n}{m}$  for multiplying Extreams and Means  $\frac{a^n}{m} = d p \times$ .

To find the Proportion of  $a$  to  $z$  from  $\frac{a^n}{y} = \frac{b z}{x}$ , here  $a : z :: \frac{b}{x} : \frac{n}{y}$ .

To find the Proportion of  $ab$  to  $d$ , from  $ab$  or  $1 ab = d n y$ .

Here  $ab : d :: n y : 1$ , for multiplying Extreams and Means we have  $ab = d n y$ .

78. I shall now show the Learner, the Certainty of the Rules on which this Science is founded; this I have purposely omitted in the Beginning of the Work, imagining it unreasonable to expect a Learner to see the Force of a Demonstration in *Algebra*, before he is acquainted with its Characters and Language.

### *The Foundation of transposing Quantities.*

THIS is grounded on the obvious Truth, *that every Thing is equal to itself*; that is,  $m = m$ , and  $-y = -y$ ; whence to transpose any Quantity, is only to make that Quantity equal to itself, prefixing to it the contrary Sign, and adding it to the given Equation. Suppose there is given

	1	$a - b + d - m = z$ , to transpose, $b, d$ , and $m$ .
Now	2	$b = b$
1 + 2	3	$a + d - m = z + b$
And	4	$-d = -d$
3 + 4	5	$a - m = z + b - d$
Lastly	6	$m = m$
5 + 6	7	$a = z + b - d + m$

### SUBTRACTION.

I say to subtract a *negative* Quantity from a *positive*, is only to change the Sign of the *negative* Quantity, and add it to the *positive*

positive Quantity, and this Sum will be the Remainder required.

That is,

If  $x - y$  is subtracted from  $x + y$ , I say the Remainder is  $2y$ , for

$$\begin{array}{l|l} \text{Suppose} & 1 \quad | \quad x + y = m \\ \text{And} & 2 \quad | \quad x - y = n \end{array}$$

Now if it can be proved that  $2y$  is equal to the Difference between  $m$  and  $n$ , it follows that to subtract a negative Quantity is to change its Sign and add it.

$$2 + y \quad | \quad 3 \quad | \quad x = n + y$$

Now in the first Equation for  $x$  write  $n + y$ , for that is equal to  $x$ .

$$\begin{array}{l|l} \text{Then} & 4 \quad | \quad n + 2y = m \\ 4 - n & 5 \quad | \quad 2y = m - n. \quad \text{Q. E. D.} \end{array}$$

I say further, that to subtract a negative Quantity from a negative Quantity, is done by changing the Sign of the Quantity to be subtracted, and then adding them by the Rules in Addition, and the Sum will be the Difference required.

$$\begin{array}{l|l} \text{Suppose} & 1 \quad | \quad x - 2y = m \\ \text{And} & 2 \quad | \quad x - y = n \end{array}$$

Now the second Equation being subtracted from the first according to the Rule, leaves  $-y = m - n$ ; and if it can be proved that  $-y = m - n$ , then to subtract a negative Quantity from a negative Quantity, is only to change its Sign and add it.

$$\begin{array}{l|l} 2 + y & 3 \quad | \quad x = n + y \\ 1 + 3 & 4 \quad | \quad n + y - 2y = m \\ 5 - n & 5 \quad | \quad y - 2y = m - n \\ \text{That is} & 6 \quad | \quad -y = m - n. \quad \text{Q. E. D.} \end{array}$$

And that  $m - n$  is a negative Quantity is evident, for  $x - 2y$  cannot be so great as  $x - y$ , they being supposed positive Quantities, and therefore  $m$  cannot be so great as  $n$ ; consequently

frequently  $m - n$  is a *negative* Quantity, and therefore may be equal to  $-y$ . And in

### MULTIPLICATION,

I say unlike Signs being multiplied give  $-$  in the Product; that is,  $-a \times a = -aa$ .

To prove which, I take for granted the following

### LEMMA.

That no Quantities connected by the Sign  $+$  only, or by the Sign  $-$  only, can be equal to *nothing*. That is, it cannot be  $-a - b = 0$ , or  $a + b = 0$ , though it may be  $a - b = 0$ , or  $b - a = 0$ .

Now, if possible, let  $-a \times a$  produce  $aa$  where the Sign of the Product is *affirmative*.

Let  $\begin{array}{l|l} 1 & m - a = 0 \\ 2 & a = a, \text{ that is, every Quantity is equal to itself.} \end{array}$   
 $1 \times 2 \mid 3 \mid ma + aa = 0a$ , by the Supposition, that is,  $ma + aa$  is equal to *nothing*, which is against the Lemma, therefore  $-a \times a$  cannot produce  $aa$ .

But, I say  $-a \times a = -aa$ .

Let  $\begin{array}{l|l} 1 & m - a = 0 \\ 2 & +a = +a, \text{ for any Quantity is equal to itself.} \end{array}$   
 $1 \times 2 \mid 3 \mid ma - aa = 0a$ , that is,  $ma - aa$  is equal to *nothing*, whence  $ma = aa$ . Now that  $ma = aa$  is evident, for  $m - a = 0$ , therefore  $m = a$ , and multiplying by  $a$  we have  $ma = aa$ , consequently  $-a \times a = -aa$ . Q. E. D.

I say further, that like Signs tho'  $-$  being multiplied, produce  $+$  in the Product. That is,  $-a \times -a$  produces  $aa$ , and not  $-aa$ , for

Let  $\begin{array}{l|l} 1 & m - a = 0 \\ 2 & -a = -a, \text{ every negative Quantity being equal to itself.} \end{array}$

Now, if possible, let  $-a \times -a$  produce  $-aa$ . Then

$1 \times 2 \mid 3 \mid -ma - aa = -0a$ , by the Supposition, that is,  $-ma - aa$  is equal to *nothing*, which is against the



the *Lemma*, therefore  $-a \times -a$  cannot produce  $-a a$ , but the Sign must be  $+$  or *affirmative*; which may be further proved thus,

Let 
$$\begin{array}{l|l|l} & 1 & m - a = 0 \\ & 2 & -a = -a \\ 1 \times 2 & 3 & -ma + aa = 0 a, \text{ that is, } -ma + aa \end{array}$$
 is equal to *nothing*, from whence  $ma = aa$ . And that  $ma = aa$  is evident, for  $m - a = 0$ , therefore  $m = a$ , and multiplying by  $a$ , we have  $ma = aa$ . Hence  $-a \times -a = aa$ . Q. E. D.

# DIVISION.

As unlike Signs in Multiplication produce  $-$  in the Product, I say that,

In Division, unlike Signs being divided, give  $-$  in the Quotient, that is, if  $ab - bb = 0$ , and both Sides of the Equation be divided by  $b$ , I say the Quotient will be  $a - b$ , and not  $a + b$ .

Suppose unlike Signs to give  $+$  in the Quotient.

If 
$$\begin{array}{l|l|l} & 1 & ab - bb = 0 \\ \text{Let} & 2 & b = b \\ 1 \div 2 & 3 & a + b = \frac{0}{b}, \text{ by the Supposition, that is} \end{array}$$
  $a + b$  is equal to *nothing*, which is against the *Lemma*, therefore an Absurdity follows the Supposition, that unlike Signs give  $+$  in the Quotient; but I say unlike Signs give  $-$  in the Quotient.

Let 
$$\begin{array}{l|l|l} & 1 & ab - bb = 0 \\ & 2 & b = b \\ 1 \div 2 & 3 & a - b = \frac{0}{b}, \text{ that is, } a - b \text{ is equal to} \end{array}$$
 *nothing*, whence  $a = b$ , and that  $a = b$  is thus proved.

$$\begin{array}{l|l|l} 1 + bb & 4 & ab = bb \\ 4 \div b & 5 & a = b. \text{ Q. E. D.} \end{array}$$

I say further, that like Signs being divided, though they are *negative*, give  $+$  in the Quotient, that is,  $ab - bb$  divided by  $-b$ , the Quotient is  $-a + b$ , and not  $-a - b$ .

If

If like Signs though — give — in the Quotient ; then

$$\begin{array}{l|l|l} \text{Let} & 1 & ab - bb = 0 \\ & 2 & -b = -b \\ 1 \div 2 & 3 & -a - b = \frac{0}{b}, \text{ by the Supposition, that} \end{array}$$

is,  $-a - b$  is equal to *nothing*, which is against the *Lemma*, therefore an Absurdity follows the Supposition, that like Signs though — give — in the Quotient.

But, I say,  $ab - bb$  divided by  $-b$ , the Quotient is  $-a + b$ , that is, like Signs though — give + in the Quotient. For,

$$\begin{array}{l|l|l} \text{Let} & 1 & ab - bb = 0 \\ & 2 & -b = -b \\ 1 \div 2 & 3 & -a + b = \frac{0}{b}, \text{ that is, } -a + b \text{ is} \end{array}$$

equal to *nothing*, whence  $b = a$ , and that  $b = a$  is evident, thus,

$$\begin{array}{l|l|l} 1 + bb & 4 & ab = bb \\ 4 \div b & 5 & a = b. \quad \text{Q. E. D.} \end{array}$$

By this the Learner will see that like Signs though — both in Multiplication and Division, must give + in the Product and Quotient, for an Absurdity follows the contrary Hypothesis, or Supposition, of their producing — in either the Product or Quotient.

The other Principles of this Science are very obvious, being the plain Consequences of the *Axioms* mentioned in the Beginning of the Work.

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